SOME STUDIES ON STATISTICAL ASPECTS OF SIZE EFFECTS ON STRENGTH AND FRACTURE BEHAVIOUR OF MATERIALS AND FRACTURE RESISTANT DESIGN

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CERTIFICATE

This is to certify that the thesis entitled 'Some Studies on Statistical Aspects of Size Effects on Strength and Fracture Behaviour of Materials and Fracture Resistant Design' by C.V.S. Kameswara Rao, for the award of the Degree of Doctor of Philosophy, of the Indian Institute of Technology, Kanpur is record of bonafide research work carried out by him under my supervision and guidance. The thesis work, in my opinion, reached the standard fulfilling the requirements for the Doctor of Philosophy Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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The classical approach of continuum mechanics of materials is inadequate to account for the phenomena like size effects on strength, ductile-brittle fracture transition etc. This is because the approach does not consider the existence of inherent structural defects like flaws in materials. Applying the theories of fracture mechanics and composite materials, the concepts of classical mechanics of materials are generalized to interpret size effects on strength, stiffness and ductile-brittle transition. Also, revised design methodologies are evolved to ensure certain levels of safety from brittle fracture, with suitable optimality criteria. The present thesis is a contribution to the aforementioned areas.

A critical review of the studies on size effects

on strength of various materials and statistical theories of strength based on weakest link concept and their application to various materials, is made. Following the observations in the review, it is emphasized that fitting test data suitable to an *a priori* chosen form of strength distribution function, as is done in present approaches, is limited in its application to a restricted class of materials and is at best an approximation. Therefore, a more general approach is proposed in this work by which a strength distribution function can be obtained corresponding to an actual form of size-mean strength relation (obtained experimentally) of a particular material. Using this formulation and method of solution, it is shown that Weibull's distribution function can be obtained as a particular case corresponding to a particular form of size-mean strength relation. Construction of strength distribution functions corresponding to more general forms of size-mean strength relations needs resort to numerical methods which are outlined in this investigation. The approach developed herein to study size effects on strength of flaw sensitive materials is motivated by the work of Tsai and Kolsky on the fracture strength of glass plates as affected by indentor diameter, by observing the anologous nature of both the problems.

The present formulation gives an alternative approach of accurately characterizing the size dependent scatter in strength of materials, although leading to mathematical complexity in computation for materials with complicated size-mean strength relations.

Idealizing cement concrete as a material that fits the weakest link concept, an attempt is made to characterize the scatter in direct tensile strength by distribution functions using the results obtained in the thesis. The experimental results of Kadleček and Špetla have been used in this study.

As an application of statistical aspects of strength in materials testing and specifying strength, the minimum number of test specimens required to be tested to predict the mean strength with a given reliability as affected by specimen size, is investigated for cement concrete. Since the coefficient of variation of strength decreases with increase in specimen size, it is shown that more specimens of smaller size are to be tested for a given reliability in predicting mean strength, than for large sizes. A study of optimum specimen size and sample number so that total cost of testing is minimum is also carried out.

A unified theory is proposed which correlates the

size effects on strength, stiffness, fracture mode transition and nonlinearity in materials like concrete and fiber reinforced mortars in which fracture initiation and total failure are not identical (due to the presence of microcracking). Consequently, the weakest link concept is not applicable. The concept used is, as the applied stress level increases, microcracks are developed which cause a progressive breakdown in internal structure resulting in the decrease of stiffness. To quantify the phenomenon, the probability of failure of the material is related to the probable number of microcracks in the material. Furthermore, to estimate the loss in stiffness with increasing stress, an approximate expression for the elastic modulus of a composite material in which one of the phases happens to be voids is related to the micro-crack volume through a semi-empirical constant. Since the probability of failure is related to the applied stress level as well as the size, the proposed theory yields a size and stress dependent value of the effective Young's modulus, which reflects the nonlinearity in stress strain behaviour and the size effect. The proposed theory also brings to light a new phenomenon viz, the size effect on stiffness and interprets the size effect on ductile-brittle transition from a different view point from that of Glucklich. The predictions of the theory,

i.e. the lower stiffness and lower ductility associated with specimens of larger size are shown to be qualitatively in agreement with observations in tests conducted on fibre reinforced cement mortar briquettes.

The fracture resistant design with emphasis on material choice, of a pressure vessel in the presence of a part-through crack is attempted. The structural requirement is that the vessel should withstand the internal pressure cycles. The service life is estimated by considering the crack growth due to internal pressure cycles. In the problem solved, six different materials are considered and the material choice is made so as to maximize the utility which is defined as the net monetary return considering service life, initial cost and cost incurred in service because of the weight.

Problems for further investigations are suggested as relevant to those discussed within the scope of the thesis.

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NOTATIONS

A = cross sectional area;

A_1 = constant relating fracture stress to crack length;

A_6 = cross sectional area of a 6 in. cube;

a = constant in Bolotin's size - mean strength relation; also depth of part - through crack;

$a_1, a_2, \ldots a_{n+1}$ = roots of the equation $V\, D(V) = 0$;

a_{cr} = critical depth of part - through crack;

a_{final} = final crack depth (critical);

$a_{initial}$ = initial crack depth (critical);

B = stress dependent risk of rupture;

b = constant in Bolotin's size - mean strength relation;

C = crushing stress; also material cost of pressure vessel in Rs. per inch length;

C_T = total cost of testing in Rs.

C_V = coefficient of variation;

c = half crack length; also semimajor axis of part - through crack;

$\dot{c}$ = crack velocity;

c_{final} = half final length of major axis of crack (critical);

$c_{initial}$ = half initial length of major axis of crack (critical);

c_k = material constants in the distribution function of Greene;

c_o = effective fractional void volume;

c_1 = constant in size - mean strength relation;

c_2 = constant in the extreme value distribution;

$\bar{c}$ = effective fractional microcrack or void volume;

$\bar{c}_1$ = effective fractional microcrack or void volume in a specimen of size V_1;

$\bar{c}_2$ = effective fractional microcrack or void volume in a specimen of size V_2;

D = diameter of the pressure vessel;

D(V) = polinomial in V of degree 'n';

$\bar{D}(V)$ = V D(V);

$\bar{D}'(.)$ = derivative of the function $\bar{D}$ evaluated at (.);

d = lateral dimension of a specimen, diameter;

d_6 = lateral dimension of a 6 in. cube;

E = Young's modulus; also maximum percentage error of the sample average;

E^* = effective Young's modulus;

F(.) = double exponential distribution of the argument (.) in the theory of Kase;

$F_V(R_1)$ = cumulative probability of failure upto a stress level R_1 of a specimen of size V;

F_o = fracture stress;

F_1 = distribution of smallest values;

f(x) = probability density function of flaw length x;

$f(\sigma)$	= probability that fracture occurs for an applied stress level lying between σ and $\sigma + d\sigma$;
$\bar{f}(\sigma)$	= function of σ increasing with σ and satisfying suitable defined conditions;
f_o	= applied force;
$f_1, f_2, \ldots f_5$	= functions characterizing various mechanisms of energy dissipation;
$f^1_{ij}, f^2_{ij}, f^3_{ij}$	= functions of θ characterizing crack tip stress field;
G	= energy release rate; also shear modulus of solid phase;
$G(\sigma)$	= the integral $\int_{\sigma_L}^{\sigma} \phi(\sigma)\, d\sigma$;
G_c	= critical strain energy release rate;
G_i	= shear modulus of the i th phase;
G^*	= effective shear modulus;
$H(.)$	= Heaviside unit step function of argument (.);
h	= height of specimen;
h_1	= chosen crack length in the theory of Fisher and Hollomon;
I_m	= the integral $\int_0^{\infty} \exp(-Z^m)\, dZ$;
K	= material parameter in the size-mean strength relationship;
K_c	= fracture toughness;
K_1, K_2, K_3	= stress intensity factors in the respective modes of crack propagation;
K^*	= effective Bulk modulus;
K_1^*	= cost of unit volume of material in Rs.; also return in Rs. from one cycle of service of the pressure vessel;

K_2^* = cost of testing a single specimen in Rs.; also cost incurred due to weight in Rs. per cycle of service;

K_3^* = length of pressure vessel;

K = constant in size - mean strength relationship;

k_o = constant denoting the mean volume of a single microcrack;

k_1 = semi-empirical constant relating the microcrack volume to effective microcrack or void volume;

k_m = modified stress intensity factor;

$L^{-1}(.)$ = Inverse Laplace Transform of function (.);

M = spring constant in compliance method of K_c determination;

m = material parameter in Weibull's distribution function;

N = number of test specimens; also number of phases in a composite material; also number of volume elements;

N_s = service life in cycles;

$N(V)$ = polinomial in V of degree $n-1$ or less;

N_1 = total number of macroscopic volume elements in a specimen of size V_1;

N_2 = total number of macroscopic volume elements in a specimen of size V_2;

$\bar{N}$ = total number of test specimens;

n = number of defects or flaws per unit volume; also material parameter;

$n(\sigma)$ = number of elements ruptured during loading upto a stress level σ;

$n_o(\sigma)$	=	function of stress level σ ; denoting the probability of failure of an element;
n_σ	=	number of specimens ruptured upto a stress level σ ;
n_1	=	sample size; also number of elements that get ruptured during loading upto a stress level σ in a specimen of size V_1;
n_2	=	number of elements that get ruptured during loading upto a stress level in a specimen of size V_2;
n^*	=	number of macroscopic elements in the material per unit volume;
P	=	prism strength;
P(.)	=	density function denoting the probability of the variable (.) taking a value over an infinitismal interval;
P_6	=	strength of a 6 in. cube;
$P^*_{n,\alpha}(\beta)d\beta$	=	probability that a specimen containing n flaws has a strength lying between β and $\beta+d\beta$ as a function of α;
p	=	internal pressure, psi;
Q	=	crushing load; also flaw shape parameter defined in terms of $\bar{\beta}$ and ϕ ;
Q_{cr}	=	critical flaw shape parameter;
R_1	=	fracture stress;
$\bar{R}$	=	constant in Bolotin's size - mean strength relation;
$\bar{R}_o$	=	constant in Bolotin's size - mean strength relation;
r	=	distance of point of interest from origin in crack tip stress analysis;

S = cumulative probability of failure in Weibull's distribution;

S_d = standard deviation of strength;

S_o = probability of failure of an element for a stress level between 0 and σ ;

S_2 = nondimensionalized fracture stress in the theory of Fisher and Hollomon;

S_r = nondimensionalized fracture stress;

S_X, S_Y, S_Z = principal stresses along the axes X,Y,Z;

S^* = strength of material in the presence of a flaw;

S_1^* = fracture stress in the presence of a crack;

S_n^* = fracture stress of a specimen containing n flaws;

S_o^* = strength of a material in the absence of any flaw;

S_o^{**} = model strength of a specimen in the presence of flaw;

s = standard deviation of flaw strength; also parameter in Laplace transformation;

s_c = constant in Bolotin's equation;

s_u = mean value of upper yield point stress;

s_o = lower bound on fracture stress;

T = surface energy per unit area;

t = time; also denotes function $\psi(u)$; also parameter 't' in the students' 't' distribution, and also wall thickness of pressure vessel;

U = strain energy;

U_R = net monetary Return in Rs.

u	=	variable of integration;
V	=	specimen size volume;
V_1, V_2, V_3	=	volumes of specimens of various sizes;
V_{opt}	=	optimum specimen volume;
V_o	=	aribitrary volume for nondimensionalizing;
v	=	volume fraction of voids;
v_o	=	volume of microcracks;
$\bar{v}$	=	effective volume of microcracks or voids;
$v_{failure}$	=	microcrack volume at failure;
v_i	=	volume fraction of the i th phase;
$\bar{v}_1$	=	effective volume of microcracks or voids in a specimen of size V_1;
$\bar{v}_2$	=	effective volume microcracks or voids in a specimen of size V_2;
W	=	surface energy;
w	=	weight of pressure vessel in lbs. per inch length;
X	=	random variable; also a coordinate in rectangular coordinate system;
x	=	flaw strength;
x_o	=	lower bound on the value of random variable;
x_1	=	minimum value of the random variable; also flaw size;
x^*	=	most probable flaw size;
X, Y, Z	=	orthogonal coordinates;
α	=	constant in extreme value distribution; also denotes the ratio of lateral stress to axial stress in a triaxial stress field;
α_o	=	limit to which 't' tends as u tends to ∞

α^*	=	constant denoting decrease in material strength with increase in flaw size; also constant defined in terms of effective Poisson's ratio ν^* ;
β	=	non dimensionalized stress along 'Z' axis; also crack propagation factor;
β^*	=	constant defined in terms of effective Poisson's ratio ν^* :
$\bar{\beta}$	=	ratio of applied stress to yield stress;
$\Gamma(.)$	=	Gamma function of argument (.);
$\bar{\Delta}$	=	space average dilatation;
ΔV_i	=	i th elemental volume;
δ	=	any parameter (like, water - cement ratio) on which energy dissipation depends;
ϵ_u	=	strain at necking instability;
$\bar{\epsilon}_{ij}$	=	space average strain tensor;
η	=	nondimensionalized quantity as defined in the theory of Kase, denotes the deviation of strength from modal strength;
θ	=	angular coordinate of the point of interest in crack tip stress analysis;
λ	=	constant in the exponential distribution;
λ^*	=	effective Lame's parameter;
μ	=	mean flaw strength in the theory of Frenkel and Kontorova;
μ^*	=	effective Lame's parameter;
ν	=	Poisson's ratio;
ν^*	=	effective Poisson's ratio;
ξ	=	dummy variable of integration;

π	= function defined in terms of nondimensionalized fracture stress in the theory of Fisher and Hollomon; also denotes infinite product;
σ	= applied stress;
σ_g	= nominal peak stress;
σ_{ij}	= stress tensor;
σ_L	= lowest mean strength of material;
σ_m	= modal strength;
σ_U	= ultimate stress;
σ_u	= material parameter in Weibull's distribution;
σ_y	= yield stress;
σ_o	= material parameter in Weibull's distribution;
$\bar{\sigma}$	= mean failure stress;
$\bar{\sigma}_b$	= mean failure stress in bending;
$\bar{\sigma}_{ij}$	= space average stress tensor;
$\bar{\sigma}_t$	= mean failure stress in tension;
$\bar{\sigma}_u$	= upper yield point stress in flexure;
$\bar{\sigma}_1$	= mean strength of specimen of size V_1;
$\bar{\sigma}_2$	= mean strength of specimen of size V_2;
σ^*	= Griffith's fracture stress;
$\Phi(\sigma)$	= probability that fracture does not occur for a stress lying between 0 and σ;
$\Phi^*(\sigma)$	= probability that fracture occurs for a stress lying between 0 and σ;
$\Phi_o(x)$	= probability density function of flaw strength x;

$\Phi_1^*(\sigma)$	=	cumulative probability of failure of a specimen of size V_1;
$\Phi_2^*(\sigma)$	=	cumulative probability of failure of a specimen of size V_2;
ϕ	=	elliptic integral of the second kind;
$\phi_o(\alpha)$	=	function defined in terms of material parameter α;
$\{\phi(\sigma)d\sigma\}\Delta V$	=	probability that a potential flaw exists in an elemental volume ΔV and is of such size that fracture occurs when the applied stress is between σ and $\sigma + d\sigma$;
$\psi(u)$	=	function defined so that $\psi'(u) = \phi(u)$.

CHAPTER ONE

INTRODUCTION

1.1 GENERAL

The search for improved knowledge of mechanics of materials and new design philosophies appears to be a continuous process. However, the stringent needs of modern technology have emphasized on the need for refined understanding of mechanics of materials in order to reduce the gap between apparent strength and theoretical strength (based on atomic structure and nature of bond), new design methodologies, and development of new materials. The basic feature that motivated study into all the above areas of investigation, is essentially a tendency to move closer to reality with minimum of empiricism and uncertainity. While the earlier attempts of engineering design are motivated towards use of already existing materials, the current design methods focus on the need for examination of material - structure interaction during the design process (1,2,3,4) as well as the design of new materials. In addition, the conventional approaches of engineering design do not reflect explicit measures of design efficiency like optimality with respect to weight or cost. However, the stringent demands of economy, of

late, require a comparative study of alternative design solutions. Consequently, the importance of several optimality criteria and cost-effectiveness models has been recognized (5,6,7,8).

The search for rationality in the concepts of structural safety (9,10,11,12,13) and the observation of the randomness in material properties and loads have paved the way for considering the reliability as a measure of safety. As a consequence, several attempts are being made to examine the probabilistic aspects of strength of materials and their testing. Of related interest to the aforementioned areas is the phenomenon of size effects on material behaviour and associated scatter in properties of materials. The importance of size effects on material behaviour becomes obvious in any attempt to correlate the observations on strength and failure mode noticed from simple tests to those in a full scale structure. While the earlier studies in scientific design processes did not warrant a thorough examination of size effects on material behaviour, these cannot be neglected in any rational investigation. This is especially true of ceramic type materials (for eg: clay, cements, gypsum, concretes, porcelain etc.) which show brittle fracture characteristics and strength as

a function of size. This can be attributed to the presence of flaws or internal cracks during their manufacturing process. Several studies on the size effects on strength of materials have brought to light the importance of defects and flaws in materials and have paved the way for new disciplines like fracture mechanics. As a result of the application of the concepts of fracture mechanics several phenomena like brittle fracture, fatigue and creep rupture (14,15,16) can be reasonably interpreted. In addition, fracture resistant structural design can be more rationally tackled with the use of concepts of fracture mechanics. Eventhough the achievements hitherto, in the aforementioned areas like material - structure design, rational concepts of safety, statistical aspects of strength of materials, standardization of testing methods etc. are quite promising, further investigations are essential for a better understanding of their nature and application. The present thesis is an attempt to investigate some of the aforementioned areas and to synthesize the current state of knowledge with the current investigation on brittle fracture statistics.

In the following sections of this chapter, the salient results of various studies into the areas of

fracture mechanics, size effects on strength and behaviour of materials and fracture resistant design are reviewed while the statistical strength theories are critically reviewed in Chapter 2. The presentation of the review provides a basis and motivation for specific problems that are dealt with in the subsequent chapters of the thesis.

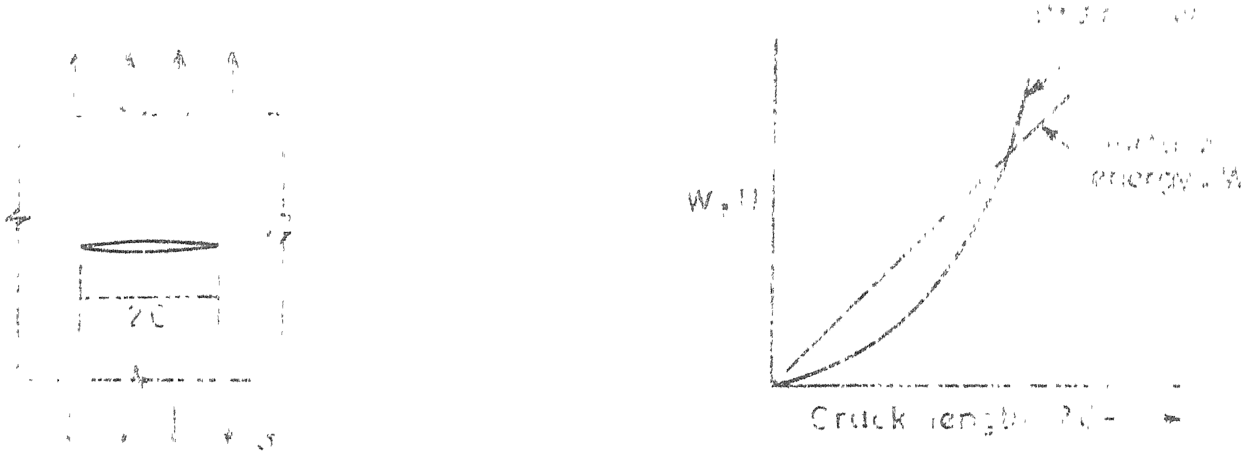
The effect of specimen size on strength and mechanical behaviour of materials is of importance in engineering design, specification of strength of materials as well as in correlating observations from tests on models to those in the prototype. Some materials of engineering interest are more flaw sensitive than others. Fracture mechanics provides a basis for the study of flaw sensitive materials. Eventhough flaws in materials may be randomly distributed and interaction exists between various flaws, in fracture mechanics, only the equivalent potential flaw or critical flaw and its effect on strength is usually studied. Qualitatively, the larger the specimen size, greater is the probability of having an effective critical flaw in the material and consequently lower the mean strength. Consequently the topic of fracture mechanics provides the starting point for various

statistical strength theories. In the following sections the basic concepts of fracture mechanics and its application to engineering design are briefly discussed.

1.2 FRACTURE MECHANICS

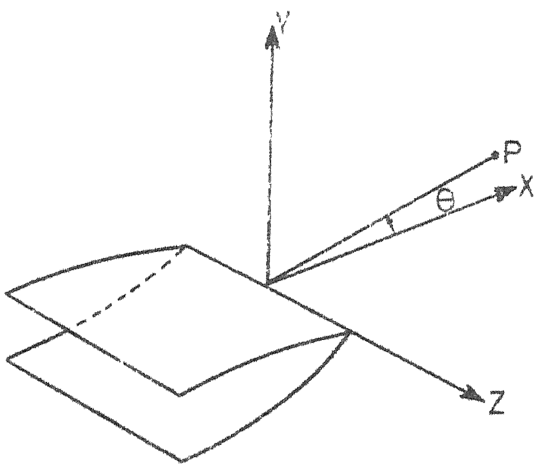
The need for a proper recognition of the presence of crack like defects or flaws in materials is felt in the wake of several studies on strength of materials as well as sudden catastrophic fractures in several structures (Refer Sections 1.2.1 and 1.3 of the thesis). The mechanism of fracture of materials in the presence of crack like defects is termed as fracture mechanics. The subject provides a basis for several statistical strength theories and fracture resistant design methods, as will be discussed later.

The origin of the evolution of the subject of fracture mechanics can be traced back to the work of Griffith (17). Griffith reasoned out that the material strength at rupture could be significantly affected by the presence of crack like defects in a material. Making use of the elastic stress field in an infinite tension panel in the presence of an elliptical crack (Fig. 1.1) as given by Inglis (18) and using the energy balance considerations at the onset of crack propagation, Griffith advanced the condition that 'unstable crack



Uniaxial tension in the presence of a crack

FIG. 1.1 GRIFFITH'S FRACTURE CRITERION.



... E SYSTEM FROM THE LEADING EDGE OF CRACK.

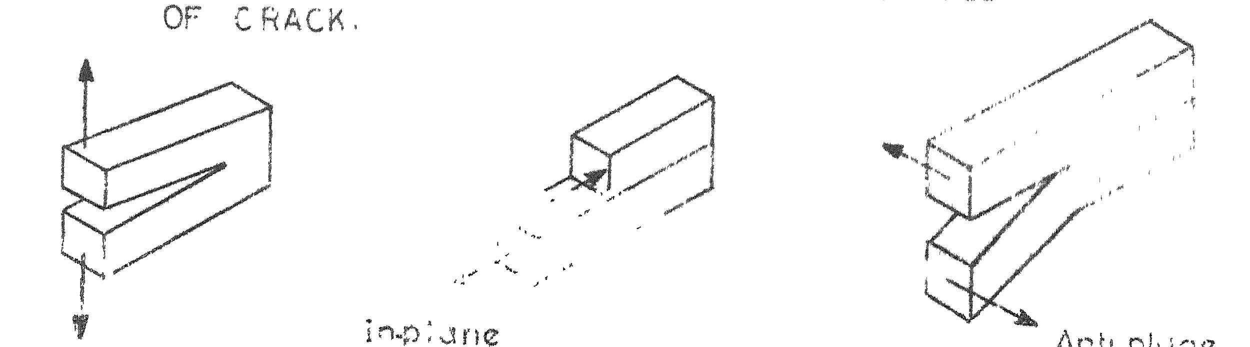


FIG. 1.2 COORDINATE SYSTEM FOR CRACK TIP STRESSES AND MODES OF CRACK PROPAGATION.

propagation occurs when the elastic strain energy release rate with respect to crack length from the medium is at least equal to that of the surface energy absorbed by the newly formed crack surface. This can be written in the following form,

$$\frac{\partial U}{\partial c} \geq \frac{\partial W}{\partial c} \qquad .. (1.1)$$

where U = the strain energy

W = the surface energy

and 2c = the crack length

Eqn. 1.1 on substitution of the relevant quantities for linear elastic case takes the form

$$\sigma^* = \sqrt{\frac{2ET}{\pi c}} \qquad .. (1.2)$$

where σ^* = the failure stress in the presence of crack of length 2c

E = Young's modulus

and T = the surface energy per unit area

Eqn. 1.1 means that fracture occurs when the term on the left hand side of Eqn. 1.1 attains a particular value called 'Critical Strain Energy Release Rate' denoted by G_c. Griffith's energy balance criterion for fracture as given by Eqn. 1.1 is applicable only to

ideally brittle materials which behave elastically, in which the strain energy is completely reversibly stored upto the onset of fracture and no other energy dissipating mechanism other than surface energy is present. It is suggested by Irwin (19) and Orowan (20) that the range of applicability of Griffith's criterion could be extended by considering the energy dissipated in the form of plastic deformation along with the surface energy. As remarked by Narayan Swamy (21), a very general form of energy balance criterion for crack propagation can be written in the form

$$\Delta U = \Delta\{f_1(c) + f_2(c, \sigma, t, \delta) + f_3(c^2, \sigma^2, E^2, \dot{c}^2) - f_4(c) - f_5(c^2, \sigma^2)\} \qquad .. (1.3)$$

where f_1 = surface energy, function of crack length c

f_2 = time dependent irrecoverable deformation, function of crack length c, stress level σ, time t and other parameters δ

f_3 = kinetic energy of crack propagation dependent on $\dot{c}$, the crack velocity

f_4 = energy at stress concentrations

f_5 = energy of the applied stress field.

Inspite of the fact that Griffith's work brought to light the significant effect of the presence of fortuitious flaws on material strength, it remained purely to be of theoretical value till the real engineering theory of fracture mechanics was developed by Irwin as outlined below (22). The lack of motivation to seek the applicability of Griffith's theory to engineering problems and design, appears to be due to the fact that the quantity, critical strain energy release rate G_c, lacks a direct physical meaning and also that the surface energy cannot easily be measured.

Before going into work of Irwin and the evolution of the engineering fracture mechanics, it is necessary to mention the contribution of Sneddon (23). Sneddon considered the elastic analysis and the related Griffith problem of a penny shaped crack and helped to disclose the fact that the elastic stress field in the neighbourhood of a crack can be represented by the stress components σ_{ij},

$$\sigma_{ij} = \frac{1}{\sqrt{r}} \{ K_1 f^1_{ij} (\theta) + K_2 f^2_{ij} (\theta) + K_3 f^3_{ij} (\theta) \}$$

$$+ \ldots \text{ other nonsingular terms} \qquad \ldots (1.4)$$

where K_1, K_2, K_3 = stress intensity factors of the dimension psi $\sqrt{\ }$in

r, θ = coordinates characterizing the location of the point at which stress field is sought (Fig. 1.2)

$f^1_{ij}, f^2_{ij}, f^3_{ij}$ = scalar function of θ.

Eqn. 1.4 reflects reasonably the so called crack tip singularity and K_1, K_2, K_3 are consequently known as stress intensity factors which are dependent on the Young's modulus and crack length. Evaluation of K_1, K_2, K_3 in any particular problem is adequate to characterize the stress field around the crack tip as far as fracture problems are concerned. Sneddon's work not only stimulated active research in elastic analysis of crack problems by identifying them to fall in a special class of boundary value problems, but also paved the way for the origin of several physically meaningful results in terms of stress intensity factors. An excellent account of the crack problems in elasticity is given by Sneddon and Lowengrub (24). The effect of crack tip cohesive forces on the stress intensity factors is studied by Barenblatt (26).

Following the works of Griffith (17) and Sneddon (23), Irwin (22) brought out the physical significance of the stress intensity factors. Irwin related these to the concept of critical strain energy release -

rate, as follows. Considering the case of Griffith's problem, Irwin argued that, by superimposing a self equilibrating tensile stress on a free crack surface over a length interval (starting from the crack tip), the crack can be closed on the same length interval and the force required to do so is the same as the strain energy release rate. In other words, the strain energy release rate is the same as the crack extension force and there exists a critical value of the stress intensity factor K_c called Fracture Toughness corresponding to G_c. A quantitative evaluation of the above arguments leads to the relationships

$$\begin{aligned} K_c &= \sqrt{E\, G_c} && \text{(Plane stress)} \\ &= \sqrt{\frac{E\, G_c}{1-\nu^2}} && \text{(Plane strain)} \end{aligned} \qquad \text{.. (1.5)}$$

where ν is the Poisson's ratio.

The most general mode of crack propagation in a three dimensional medium can be seperated into three independent modes of crack propagation (Fig. 1.2), namely the tensile mode, the in-plane shear mode and the antiplane shear mode. Under general loading conditions and crack geometries, the three stress intensity factors K_1, K_2 and K_3 are adequate to

characterize the three modes of crack propagation. A comprehensive catalogue of the values of stress intensity factors corresponding to various crack geometries and load conditions is compiled by Paris and Sih (25). The fracture toughness of a material can be determined either by employing cracked test specimens (27) or by the compliance method (28,29). In the former, a specimen with known crack geometry is loaded and the load and crack length at failure are noted and the fracture toughness is obtained through the corresponding stress intensity factor. In the compliance method the spring constant M of a cracked specimen is noted corresponding to various values of crack length c and the strain energy release rate is given by

$$G = \frac{\partial U}{\partial c} = \frac{1}{2} f_o^2 \frac{\partial}{\partial c} \left(\frac{1}{M}\right) \qquad .. \ (1.6)$$

where G = strain energy release rate

U = strain energy

f_o = applied force

Knowing G from Eqn. 1.6 at the onset of fracture, G_c is obtained and K_c from Eqn. 1,5.

1.2.1 Role of Fracture Mechanics in Engineering Design

Several studies on structural elements like pressure vessels, ship plates, turbine rotors etc., indicated that typical brittle failures occuring at operational or service loads are originated by small cracks or crack like flaws (63,64,65). A check for fracture resistance in structural design assumes greater importance while using high strength materials since fracture toughness usually decreases with increasing strength and as such the design conditions are more rationally specified in terms of fracture stress rather than yield or ultimate stress. Of the three approaches to fracture resistant design viz, the transition temperature approach (67,68), the application of notch stresses and the fracture mechanics approach, the last mentioned one is more general in its application.

The fracture mechanics approach to design of structures is based on the criteria and conditions under which self supported crack propagation can take place as dependent on the material properties, presence of crack like imperfections, size and shape of the body and other environmental conditions. Given a design problem and the possible materials, use of fracture mechanics approach in principle consists of the

following steps. First the fracture toughness of the various materials at various temperatures is determined, the temperature ranges being relevant to the design problem. The central problem of the solution is in estimating the most likely critical crack sizes, shapes and orientation, that can occur during the service life of the structure as affected by the sustained and fatigue loads. While the crack size estimates can be made depending on the accuracy of the nondestructive testing equipment available, fairly reliable information of engineering value can be obtained from full scale testing and a study of the earlier case studies.

The next phase of the design process is an analysis of the structure for a crack size-strength relationship at various temperatures. Since the stress intensity factors as relevant to various crack shapes and loading conditions of practical interest are available (25) and various numerical techniques are possible for their ready evaluation, the aforesaid analysis poses no problem. With this information as well as economic and other design values, the material that suits best is chosen keeping an estimate of the most probable flaw size. The aforementioned procedure is essentially cursory in exposition and various design problems call for special techniques in adopting the basic principle

outlined. A detailed account of the various design practices as applied to different structures can be found in the work edited by Leibowitz (64). An integrated fracture design approach with a measure of optimality is discussed in Chapter 9 of the thesis.

1.3 SIZE EFFECTS ON STRENGTH OF MATERIALS

Classical concepts of flow and fracture of materials based on various postulates like limiting state of stress or limiting state of strain criteria are based on certain idealizations of material behaviour and as such are inadequate in rationally visualizing various anomolies normally experienced with failure behaviour of materials. This is because the flow and fracture of materials in its origin is a highly localized phenomenon basically initiated and propagated by various inherent defects or flaws or structural irregularities present in almost all materials. Defects could be both on macroscopic and microscopic scale. Lattice imperfections in crystals, dislocations and grain boundaries in metals, surface flaws in glass, microcracks and inclusions of foreign materials (like that of aggregates in concrete and minerals in rocks) are typical examples of these strength impairing defects. Eventhough the exact origins of imperfections are not clearly known

yet, these could be considered as inherent material characteristics which are originated during the process of material manufacture and subsequent treatment given. By their nature, imperfections are random in number, orientation and distribution throughout the material. The classical approaches to flow and fracture, attempt to examine the failure behaviour only on the average and as such are mathematically simple. However, such attempts cannot explain the phenomena like size effects on strength, scatter associated with testing of materials for static and fatigue conditions etc. Furthermore, the scatter associated with testing of materials, specimen size effects etc. are of significant engineering importance. This can be visualized easily by noticing that brittle materials display strength variations ranging from about 30% to 300% of their mean strength (14), even under apparently identical test conditions. Consequently, characterizing such materials by a single value called mean strength may not always be justifyable in terms of engineering application. Recognizing that materials are flaw sensitive and that flaw occurance is not a deterministic phenomenon, one is prompted to the application of mathematical theories of probability and statistics. Before going further into this aspect of the problem in the thesis, an attempt

is made in what follows to review and summarise several investigations of size effects on strength in different materials. The statistical aspects of the various studies are critically reviewed in Chapter 2.

1.3.1 Cast Iron and Mild Steel

McPherran (30) experimentally investigated the effect of increased cross section size on the tensile strength of cast iron. Three types of the material were employed, in each case the diameter of the test specimen was the variable. The diameters were varied from 1.25 in to 4 in and intotal specimens of six sizes were tested. It was noticed that increased specimen size caused a decrease in the mean tensile strength. Campbell(31) conducted tension tests on cast iron rods of 15 in. length and of diameters 0.25 in., 0.375 in., 0.625 in., 0.875 in., and 1.2 in. The specimens were prepared from nine different heats and the results indicated a decrease in tensile strength with increased specimen size. Schneidewind and Hoenicke (32), in a very comprehensive experimental investigation have studied size effect on strength of gray cast iron. For assessing the tensile strength of the material the authors have employed test bars of sizes 0.5 in., 0.875 in., 1.2 in. and 2 in. in diameter and of length

18 in. to 24 in. The observed tensile strengths of the bars respectively were 52,267, 34,142, 32,050 and 22,523 psi. Strength of the same material in compression was studied by testing cylinders in compression with length to diameter ratio of two, and the strengths were respectively 112,342, 112,017, 119,583 and 99,783 psi which show lesser variation than that for the case of tension. The modulus of rupture of the material was studied by testing the rods of various aforesaid diameters on spans of 8 in., 12 in., 18 in. and 24 in. respectively, with centre point loading. The corresponding values of modulus of rupture were 98,950; 75,933; 70,783 and 62,700 psi. All the above investigators have concluded that the size effect on the observed behaviour is essentially because of the effect of bulk of material on the rate of cooling from molten state to room temperature in the preparation of the test specimens and the consequent change in the internal metallurgical structure of the material.

Davidenkov et.al (33), in their study on size effect on brittle strength of steel in liquid air atmosphere noticed that the mean strength and dispersion decreased with increase in specimen size. Further the variation of strength and dispersion was adequately

explained by Weibull's theory. Richards (34) examined the size effect in the tension testing of mild steel with emphasis on the upper yield point. In an attempt to interpret the existence of upper and lower yield points, Richards proposed a model for mild steel, which is explained below. He considers that the material consists of two components one brittle and the other soft or ductile. Under increasing loads, the system behaves elastically upto upper yield point at which the brittle component suddenly fails and the load is taken by the soft component only. As a consequence, Richards considers that the phenomenon is governed by the same laws as that of brittle fracture, like the occurance of random inhomogenties, microcracks etc. The series of tests conducted on rods of 1/8 in., 1/2 in. and 1 1/2 in. diameter with corresponding volume ratio 1:64:1000 have shown values of upper yield point being 60,300, 55,600 and 53,550 psi. It was found that the relation

$$s_u = \frac{60.14}{V^{1/58.0}} \qquad .. (1.7)$$

where s_u = mean value of the upper yield point stress in psi

V = specimen volume in cubic inches

fits the experimental data well.

Richards (35) in a subsequent investigation of size effects on yielding of mild steel in flexure corroborated his earlier findings. He tested beams of mild steel of five different sizes, ten in each size. All the specimens of different sizes were made dimensionally similar, the ratio between the corresponding dimensions of the smallest and largest specimens being 6.3 while that between adjacent being 1.59. The results indicated that size effect was present on the upper yield point of mild steel and the following relationship

$$\bar{\sigma}_u = \frac{37000}{V^{11.7}} \qquad \text{.. (1.8)}$$

where $\bar{\sigma}_u$ = the upper yield point in flexure in psi

V = the specimen volume in^3.

adequately describes the variation of mean value of upper yield point with specimen volume.

1.3.2 Concrete and Rock

One of the earliest investigators of the size effect on compressive strength of concrete was by Gonnerman (36). Based on tests conducted on cylinders of diameters ranging from 1.5 in to 10 in. with length to diameter ratio 2, he noticed a decrease in strength with increase in size. Reagel and Willis (37) in their study of size effect on modulus of rupture of concrete

beams noticed that the variation in the modulus of rupture was negligible with increase in length or width of the beams. Modulus of rupture was found to decrease with increase in depth; the decrease in modulus of rupture was about 1.9% for each increase in depth by an in. Kellerman (38) in his investigation noticed that, the flexural strength of concrete decreased with increase in span as well as the cross sectional area. Tucker (39), in his analysis of the size effects on modulus of rupture, suggests that the weakest link concept adequately interprets the decrease in modulus of rupture with increase in span length and depth while a parallel element model is essential to interpret the fact that increase in beam width has neglible effect on modulus of rupture. Johnson (40), in his study of scaled down concrete finds that for a given scale of mix the mean compressive and tensile strength decreases with increase in specimen size, the variability being more at smaller sizes. Rajendran (41) tested concrete cubes of sizes 4 in., 6 in., 8 in. and 10 in. in compression. In each case twelve specimens were tested and it is noticed that the strength decreased with increased specimen size. The test results indicated that taking the strength of a 4 in. cube as 100% the percentage strength of 6 in. cube was 95.62%, that of

8 in. cube 92.14% , that of 10 in. cube was 87.84% . The trend indicates that the strength decreases by about 4% for each 2 in. increase in the size starting from 4 in. cube. In an attempt to find an overall relationship between the strength of concrete and its size and shape, Neville (42) identified three parameters viz, d the maximum lateral dimension, V the volume and h/d the height to lateral dimension ratio. Making use of the test results of several investigators on cubes, cylinders and prisms, Neville proposed the following relationships:

$$\frac{P}{P_6} = 0.56 + 0.697 \frac{d}{\left(\frac{V}{6h} + h\right)} \quad .. (1.9)$$

and

$$\frac{P}{P_6} \cdot \frac{d}{d_6} = 0.8878 \left(\frac{A}{A_6}\right)^{0.4525} \quad .. (1.10)$$

where P = the strength of the specimen psi

P_6 = the strength of a 6 in cube psi

A = the cross sectional area in^2

A_6 = area of a 6 in. cube in^2

Kadleček and Špetla (43) have made an extensive study of the size and shape effects on direct tensile strength of concrete. They have tested cylinders and prisms of

different volumes and of constant length to lateral dimension ratios. The tests indicated a decrease in tensile strength and the associated scatter with increase in specimen size. The analytical results based on the data of this experimental study are discussed more in detail subsequently in the thesis. Jaeger and Cook (45) discuss available data on rocks and conclude that the tensile strength of rocks is a highly variable property and is influenced by specimen size more than any other property of rocks.

Skinner (44) studied the effect of specimen size on compressive strength of Anhydrite and noticed that the mean strength and standard deviation decrease, with increase in test specimen size.

1.3.3 <u>Glass, Alumina and Porcelain - Ceramic Materials</u>

Griffith (17) is the earliest investigator to have studied the size effect on strength of glass fibres in a formal manner. He reported that, within the limits of experimental error, the equation

$$F_o = 22,400 \frac{4.4 + d}{0.06 + d}$$

$$= 22,400 \frac{98600}{d} \qquad .. (1.11)$$

adequately reflects the effect of diameter d (inches) on failure stress (psi) F_o of the material. Eqn. 1.11 is valid within the range of diameters that have been considered in the tests viz, 0.001 in. to 4.20 in. Size effect on tensile strength of plexiglass (although

not a ceramic material), has been studied by Durelli and Parks (75) using dogbone specimens in bending and tension and the applicability of the relation

$$F_o = c \; V^{-k} \qquad .. \; (1.12)$$

where F_o = the failure stress

c and k = constants

V = the specimen volume

Greene (46) studied size effect on strength of glass by testing circular rods in flexure. The investigation showed the size effect as determined by the nature of surface polishing and the surface area of test specimens. The results of Durelli and Parks as well as those of Greene will further be discussed in the light of statistical strength theories.

Weil, Bortz and Firestone (47) studied the effects of prior thermal history, specimen finish, test temperatures, environment and specimen size on the strength of Alumina. Test specimens of dogbone shape of volumes 0.0117 cu.in., 0.0469 cu in. and 0.0977 cu in. have been used in the test programme. It is noticed that the average tensile strengths corresponding to the three sizes are 24.4×10^{-3} psi, 27×10^{-3} psi and 19.1×10^{-3} psi; while the corresponding standard

deviations are 3.7×10^{-3}, 3.6×10^{-3} and 2.7×10^{-3} psi.

Size effect on bending strength of porcelain has been studied by Weibull (48) and Salmassy et.al (49,50). Essentially the material displays decrease in strength with increase in size. Since these studies are of significant importance in the development of new statistical strength theories, these studies are discussed subsequently, in detail.

1.3.4 Other Materials

The investigations of Comben (51) on timber beams with varying depths from 8 cm to 0.5 cm indicated size effect on modulus of rupture. The variation in the value of modulus of rupture was between about 10000 psi to 13000 psi corresponding to the range of sizes of beams tested.

Baratta and Driscoll (52), during an attempt to develop a new tension testing device that minimizes the inherent "parasitic" bending stresses, have investigated the size effects on strength of graphite. Specimens of different volumes ranging from 0.0347 cu in. to 0.4352 cu in. were tested and the corresponding average tensile strengths varied from 9600 psi to 8700 psi. Also, the coefficients of variation were between 7 to 10%.

Size effects on crushing strength of coal has been investigated by Evans and Pomeroy (53). The authors have tested coal cubes of sides 0.25 in., 0.5 in., 1 in. and 2 in., the number of cubes tested being 262, 164, 62 and 23 respectively. The mean crushing strengths correspondingly were 3700 psi, 2800 psi, 2380 psi and 1820 psi. The standard deviations were 1530 psi, 840 psi and 710 psi and 750 psi respectively. Based on the above observations Evans and Pomeroy recommonded the applicability of the following formulae

$$C \propto d^{-0.32 \pm 0.02} \qquad .. (1.13)$$

$$Q \propto d^{1.68 \pm 0.02} \qquad .. (1.14)$$

where C = the crushing stress psi

d = the side of the cube, inches

and Q = the crushing load lbs.

Studies on gypsum mortar for modulus of rutpture and compressive strengths on cylinders, by White and Sabnis (56) have indicated the size effect on strength. The decrease in strength with increase in size is more pronounced in the case of specimens that were cured in an unsealed condition rather than those in sealed condition. This feature was partly attributed to the variation in the material as affected by the nature of

curing.

Test results on size effects on strength of stearic acid, plaster of paris, cotton threads as reported by Weibull (48) indicate the same trend of decrease in strength following increase in specimen size. Some of these results are subsequently discussed in the thesis.

Shaffer (54) in his study on the growth and properties of whiskers discussed the size effects on tensile strength of lithium flouride crystal and sapphire whiskers and notices that there is a significant decrease in strength with increase in crystal thickness, area as well as circumference of the whiskers.

Some empirical aspects of the size effects in various material systems with respect to strength have been reviewed in detail by Sabnis and Aroni (55). It can be seen from the aforementioned studies that the general feature of size effect on strength, is that the strength as well as the associated scatter decreases with increased specimen size. This aspect of the scatter is further examined analytically in detail subsequently.

1.4 SIZE EFFECTS ON MECHANICAL BEHAVIOUR

Of wide interest in the mechanics of materials is the size effect on strength and mechanical behaviour. While the effects of temperature, rate of loading and

state of stress on mechanical behaviour have been studied to some extent, the size effect is relatively less understood. However the size effect is important as it is essential in correlating the behaviour of test models and prototypes in engineering design in terms of performance as well as in establishing safety (or reliability) limits. If the occurance of imperfections in materials affects strength it is a natural consequence to note that the same imperfections might affect other aspects of material behaviour such as stiffness, ductility etc.

Schneidewind and Hoenicke (32), in their experimental investigation on gray cast iron in which they tested bars of diameters 0.5 in., 0.875 in., 1.2 in. and 2.0 in. have noticed that the corresponding Young's moduli E of the material are 19530000, 17570000, 13600000 and 11980000 psi respectively. The trend indicates that the Young's modulus E decreases with increased specimen size. The variation in the value of Young's modulus with specimen size is once again attributed to the effect of rate of cooling of the material and consequent variation in internal structure of the material.

In materials like concrete it is known that the specimen size affects the surface drying and consequently

the creep characteristics of the material. Troxell, Raphael and Davis (57) in their tests for creep on concrete using cylinders of diameters 6, 8 and 10 in. with length to diameter ratio 2, have noticed, creep ratio decreased with increased specimen size i.e. smaller specimens creep more than larger ones.

Several investigations on fatigue of metals indicate that fatigue strength decreases appreciably with increase in specimen size. The size effect on fatigue is attributed to the occurance of flaws and methods of manufacture (15, 58).

Of great practical importance is the fracture mode in materials as affected by specimen size. The bulk of the specimen influences the initiation and propagation of flaws in the materials and it is known that the fracture toughness of materials decreases significantly with specimen thickness (59). This is attributed to the change in state of stress from plane stress to plane strain and therefore the extent of plastic zone around the crack tip. Qualitatively, the zone of deformation will be smaller in the case of plane strain and consequently lower fracture toughness (60). Decrease in fracture toughness with increase in thickness is of importance in engineering design as it

influences the fracture mode from ductile to brittle as the thickness increases.

The size effect on nonlinearity in stress strain behaviour and ductile brittle transition is relatively studied less. Experimental and theoretical studies due to Glucklich (58,61,62) are the few available. A formal discussion of these is carried out in detail in Chapter 7 in the thesis wherein a theoretical model to explain the phenomena is developed.

1.5 OBJECT AND SCOPE OF THE THESIS

Any problem in structural design can be considered, broadly, to consist of the three choices, i) choice of configuration ii) choice of material and iii) choice of failure mode. Of basic importance in a design process involving the three choices is the knowledge of behaviour and properties of materials. Most engineering materials are flaw sensitive and the flaw occurance being usually a random phenomenon is size dependent in addition to other parameters such as temperature, state of stress, shape of specimen, environment etc. Since the basic properties of any material like strength, stiffness and mode of fracture are determined from simple tests in laboratory on small specimens and are used to predict the properties in a prototype, size effects

play a significant role in correlating the observations from simple tests to those in structures. In materials where the scatter in strength is significant because of random flaw occurance, (for example: cement concrete) specification of strength by a single quantity has no meaning and the strength has to be specified by a probability distribution function. A knowledge of the true strength distribution function is essential in rational safety specification by applying probabilistic structural design methods. Eventhough, some methods like Ang's (12,13) extended reliability method are so formulated that they are insensitive to the exact form of distribution functions involved by a synthesis of subjective and objective probabilistic and deterministic concepts, a knowledge of the true strength distribution functions is not unwanted as they enable a more refined application of the design method. A consequence of scatter in strength of materials, as being affected by size of material specimen, is a need to plan a testing programme with a decision on the number and size of specimens to be adopted in studying the properties of a material depending on the state of stress, especially for truely brittle materials.

A knowledge of the mode of fracture of a material,

ductile or brittle is necessary in predicting the behaviour of a prototype structure using the behaviour of a model. The phenomenon of size effect on ductile-brittle transition was first recognized by Glucklich (56,61,62). The importance of this phenomenon in predictive testing can hardly be over-emphasized. Theoretical and experimental investigations into this phenomenon would be helpful in assessing the material behaviour as well as in design applications.

Brittle fracture being catastrophic and sudden in nature with a consequent damage, design methods with emphasis on material choice (so that a predefined utility of the design decision is maximized) need to be studied.

As could be seen from the above discussion, the various areas of study outlined would be of theoretical as well as of practical importance. In this thesis an attempt is made to investigate some problems in the areas of statistical aspects of size effects on strength and fracture behaviour of concrete-like composite materials and fracture resistant design. The problems studied in the thesis and the arrangement of chapters in the thesis is as follows.

In Chapter 2, a critical review of the various existing statistical theories of strength of materials and the limitations are discussed in detail. It is emphasized that the practice of assuming a priori, the form of strength distribution function and adjusting the materials parameters to a particular material is inadequate in some cases. The need for an alternative approach, viz, a method of constructing a strength distribution function corresponding to a size-mean strength relationship of any material as obtained from tests, is indicated.

In Chapter 3, the formulation of the problem of size effects on material strength and scatter is carried out using the concept of weakest link theory and the random flaw occurance in materials. It is shown that the determination of the size dependent strength distribution function in materials amounts to the solution of a nonlinear integral equation that can be specialized depending on the nature of material under consideration, i.e. the size-mean strength relation. The motivation for the study and the formulation is the work of Tsai and Kolsky (87) on the effect of indentor diameter on the impact strength of glass. The correspondance between the problem of Tsai and Kolsky and that of

size effects on strength is utilized in formulation of the problem.

In Chapter 4, the method of solution of the integral equation on the lines of that of Tsai and Kolsky (87) for strength distribution function is developed in detail. A closed form solution of the equation yielding a distribution function that can be specialized to Weibull's (48) or Bolotin's (72) form is obtained. A discussion of the possible numerical approaches to solve the problem corresponding to more general size-mean strength relation is then made.

Chapter 5, deals with the characterization of direct tensile strength of concrete by a statistical distribution function. Direct tensile strength of concrete has not been examined by earlier investigators in the light of statistical strength theories. In this Chapter, making use of the experimental results of Kadleček and Špetla (43) and the theoretical results of the preceding Chapter 4 ~~of the~~ thesis, material parameters in the distribution functions are found by trial and error, and a comparison of the theoretical and experimental distribution functions is shown graphically.

In Chapter 6, the effect of specimen size on the strength and scatter in materials testing and specification of strength is considered. Making use of Student's t distribution, the minimum number of test specimens required to be tested as affected by size, to specify the mean strength with a given maximum error and reliability in prediction is discussed. Test results of Kadleček and Špetla (43), are used to illustrate the effect of specimen size on the number of test specimens (in direct tension testing of concrete) required to be tested for a given error and reliability in prediction. It is observed that the number of test specimens increases with decrease in specimen size. Since the total cost of any testing programme is dependent on the quantity of material used as well as the number of tests, a study is made of the optimum sample and specimen size so that the total cost of testing programme is a minimum.

In Chapter 7, a phenomenological theory is developed to interpret size effects on stress-strain and fracture behaviour of concrete-like composite materials in which subcritical crack growth before rupture is significant because of the presence of energy dissipating mechanisms like microcracking. The theory is developed on the concept that at increasing stress

levels, the probability of failure of materials increases and the probability of failure is related to the internal microcracking of the materials through a semiempirical constant. Using a result for the Young's modulus of a two phase materials, in which one of the phases happens to be voids, an expression for size and stress dependent Young's modulus is derived.

In Chapter 8, an evaluation of the predictions and implications of the theory developed in Chapter 7 are discussed. Since the theory correlates probability of failure, stress level and Young's modulus, it is shown that size effects and nonlinearity in stress strain behaviour are built in the development. The theory is shown to predict lower stiffness associated with specimens of larger size. Considering the energy under the stress strain curve as a measure of ductility, it is shown that the theory predicts larger specimens display smaller ductility. The implications of this theory are compared with those of Glucklich's theory of strain energy and size effect on ductile-brittle transition in materials. The predictions and implications of the proposed theory viz, lower stiffness and lower ductility of specimens of larger size are observed to be quali-

tatively in agreement with the observations in tests on fibre reinforced cement mortar briquettes tested in tension.

In Chapter 9, fracture resistant design of a pressure vessel in the presence of a part-through crack with emphasis on optimal material choice is attempted. After emphasising on the various embrittling factors in pressure vessels and the need for design for fracture resistance, a specific design problem is considered in which the internal pressure and nominal working stress level is specified. The vessel is considered to have a part-through crack in the wall of the vessel. It is required to choose a material out of six available materials so that the utility of the choice is maximum. While considering the utility, the material cost, weight and service life are taken into consideration. Two aspects of the design problem are considered viz, the safe life design and fail safe design.

In Chapter 10, a summary of the work presented in the thesis is carried out. Further applications of the significant results of the thesis are pointed. Also potential areas of further study are indicated.

CHAPTER TWO

CRITICAL REVIEW OF STATISTICAL THEORIES OF STRENGTH OF MATERIALS

2.1 INTRODUCTION

As could be observed from the investigations of various research workers on size effects on strength of materials (reviewed in Section 1.3 of the thesis) it is a common observation that specimens of smaller size have higher strength. Further, a general feature of the various studies reported, is that scatter in strength is characteristic of wide class of materials and that the mean strength approaches asymptotically a certain minimum value as the size increases. Also the scatter in strength values for smaller sizes is larger than that for bigger sizes. These features are shown schematically in Fig. 2.1.

The phenomena of size effect and scatter prompted several investigators to advance various statistical theories of strength of materials. The basic problem of statistical theories of strength is to evolve a statistical distribution function that adequately characterizes the scatter in the strength of materials. The problem in effect is to identify admissible forms

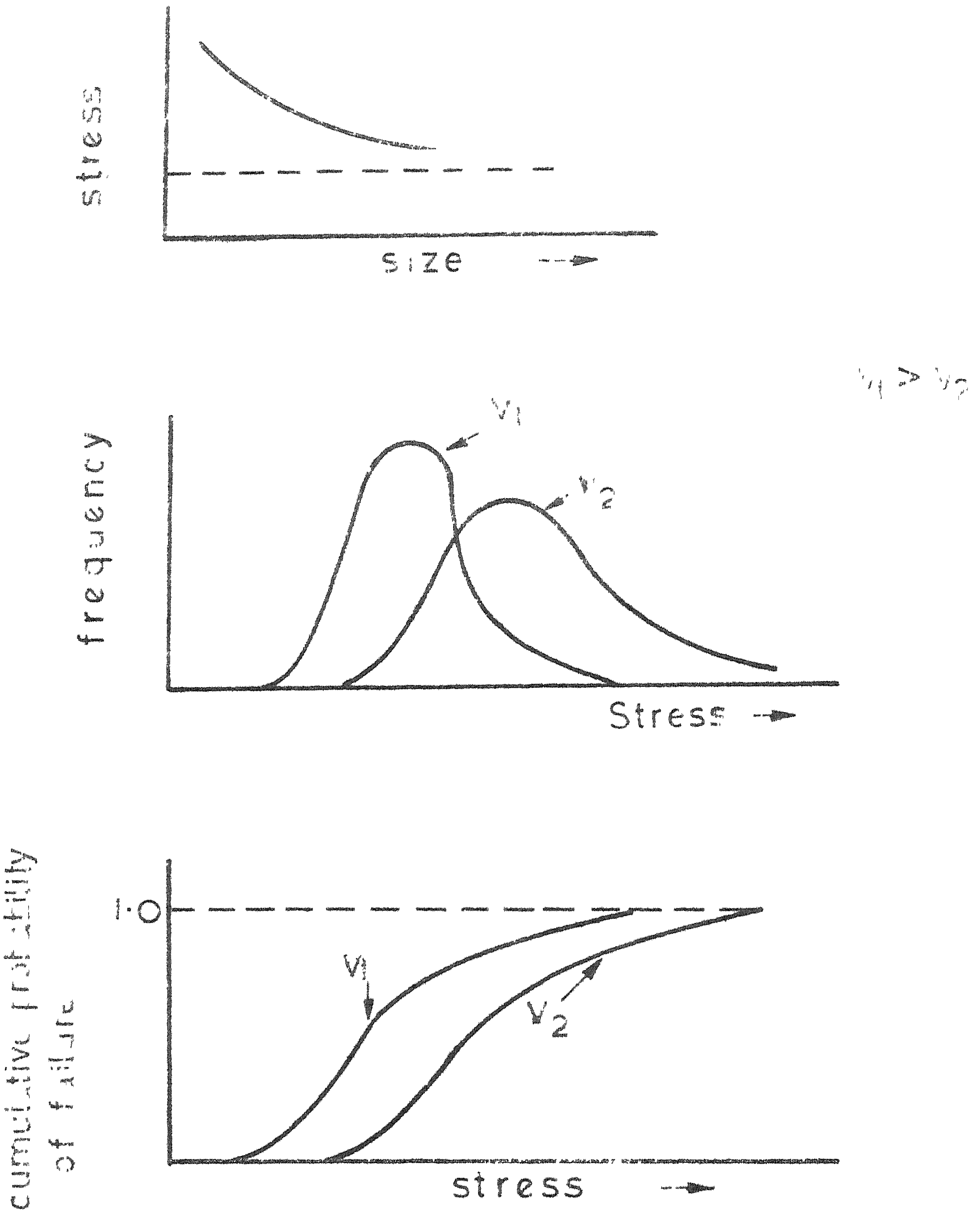


FIG.2·1 SIZE EFFECTS ON STRENGTH

of distribution functions with suitable parameters that reflect accurately the true material behaviour for which the distribution function is applied. Also, any proposed theory should be on the basis of a realistic mechanism of failure. Several theories that are developed on these guidelines are reviewed in the following. The limitations and merits of various theories are discussed subsequently at the end of this chapter.

2.1.1 Weakest link and Classical Bundle Concepts

Basically, two distinct approaches are available to study the statistical aspects of strength of materials and size effect. The approaches are based on the weakest link and classical bundle concepts (14). In the weakest link concept, the presence of a severe defect anywhere is adequate to cause total failure; that is failure of total material is same as the failure of any of the constituent elements. Consequently in this approach, strength in the large is determined by the strength of the weakest element present. On the other hand, in the classical bundle concept the strength is not determined by that of the weakest element alone but is dependent also on the strength of the elements in the neighbourhood. In this model the material is supposed to be replaced by a bundle of parallel fibers and in such a situation

the gross strength at failure is influenced by the strength of all constituents fibres. These two approaches are, in a way, idealizations to make the problem tractable and in reality the actual characteristics of materials fall in between these two idealizations. Theories based on weakest link concept are reviewed in this chapter, while those based on bundle concept can be found in the works of Daniels (69) and Fruedenthal (14). The weakest link concept has been assumed in the development made in Chapters 3,4 and 5, while a theory of composite materials which lies in between the weakest link and classical bundle concept is used in Chapters 7 and 8 for a unified theory of concrete-like materials for fracture as related to size of the specimen.

2.2 WEIBULL'S THEORY AND APPLICATIONS

2.2.1 Basic Assumptions and Development of the Theory

Weibull is one of the earliest proponents of a systematic study of the statistical aspects of strength of materials (48,70,71). Weibull pointed out the inadequecy of characterizing material strength by a single quantity as is usually done in deterministic approaches and built up a theory in which additional parameters are invoked to characterize the strength of a material. Weibull employed the weakest link concept in that he considered

the failure of the total material is the same as the failure of any one of the constituent elements or the weakest element. If the material is considered to consist of a series of elements of unit volume and if the probability of failure of the element for a stress lying between 0 and σ is S_o, then the probability of survival of the element is given by $(1-S_o)$. If S denotes the cumulative probability of failure of a specimen of total volume V, then the probability of survival of the total specimen is given by

$$(1 - S) = (1 - S_o)^V \qquad .. (2.1)$$

or $$\log (1 - S) = V \log (1 - S_o) \qquad .. (2.2)$$

Weibull defines a function B termed as 'risk of rupture' given by

$$B = -\log (1 - S) = -V \log (1 - S_o) \qquad .. (2.3)$$

The risk of rupture of the element is given by

$$dB = -\log (1 - S_o) dV \qquad .. (2.4)$$

It is reasonable to assume that S_o is some function $n_o(\sigma)$ of stress level σ so that

$$dB = n_o(\sigma)\ dV \qquad .. (2.5)$$

or $$B = \int n_o(\sigma)\ dV \qquad .. (2.6)$$

and $$(1 - S) = e^{- \int n_o(\sigma)\, dV} \quad .. (2.7)$$

so that $$S = 1 - e^{- \int n_o(\sigma)\, dV} \quad .. (2.8)$$

It can be seen that from the above expression for the probability of failure S, knowing $n_o(\sigma)$ one can readily evaluate S.

For the case of a simple uniaxial stress field, Weibull intuitively suggested the use of the form of function $n_o(\sigma)$ given by

$$n_o(\sigma) = (\frac{\sigma}{\sigma_o})^m \quad .. (2.9)$$

where m and σ_o are two additional parameters to characterize the strength distribution of the material; m is called the flaw density parameter while σ_o is called the scale effect parameter.

With this assumed form of $n_o(\sigma)$, Eqn. 2.9, S is given by

$$S = 1 - \exp \{ -V (\frac{\sigma}{\sigma_o})^m \} \quad .. (2.10)$$

It can be seen from the above equation that the specimen of volume V has a probability of failure unless the stress level is zero. This implies that the material can have the lowest strength as zero. However many brittle materials do sustain some minimum stress

However, Weibull (48), in his paper expresses $\bar{\sigma}$ in terms of an integral

$$I_m = \int_0^\infty e^{-z^m} dz \qquad \text{Equivalent to Eqn. 2.13}$$

The variance of strength S_d^2 is given by

$$S_d^2 = \sigma_o^2 V^{-2/m} \{\Gamma(1+2/m) - \Gamma^2(1+\frac{1}{m})\} \quad .. (2.14)$$

where the complete gamma function is defined as before.

The above expressions for $\bar{\sigma}$ and S_d^2 indicate that as specimen size increases the mean strength and variance decrease in agreement with the empirical observations of size effect on some materials.

Weibull further generalized his theory to multidimensional state of stress and varying stress fields. In these extensions, he neglected multiaxial stress interaction (unlike that done in strength theories for complex state of stress) and assumed that only principal tension initiates failure. In the present thesis the case of uniaxial stress field is considered and as such no attempt is made to study the extensions of Weibull for other cases.

2.2.2 Bolotin's derivation of Weibull's Distribution Function

Bolotin (72) rederived Weibull's distribution function in a more formal manner by invoking a result of the distribution of minimum values. Using the basic concepts of the extreme value statistics, Bolotin shows that the distribution $F_1(x)$ of minimum values x_1 of a random variable X, for very large sample size n_1, can be given by the following

$$F_1(x_1) = 1 - \exp\{-c_2 n_1 (x_1 - x_0)^\alpha\} \quad \text{for } x_1 > x_0$$
$$= 0 \quad \text{for } x_1 \leq x_0 \qquad ..(2.15)$$

where c_2, α = positive constants

and x_0 = the lower bound on variable X

Identifying the problem of strength theory of materials based on the weakest link concept as the same as the study of the minimum strength of a defect in a material in the presence of a population of defects, Bolotin uses the analogy to write the strength distribution function as follows

$$F_V(R_1) = 1 - \exp\{-c_2 nV (R_1 - s_0)^\alpha\} \quad \text{for } R_1 > s_0$$
$$= 0 \quad \text{for } R_1 \leq s_0$$

..(2.16)

where F_V = cumulative probability of failure at a stress level R_1 of a specimen of size V.

V = volume of the specimen

n = number of defects per unit volume

R_1 = fracture stress

s_o = lower bound on fracture stress.

Eqn. 2.16 in a nondimensional form is written as

$$F_V(R_1) = 1 - \exp \{ - \frac{V}{V_o} (\frac{R_1 - s_o}{s_c})^{\alpha} \} \quad .. (2.17)$$

where $c_2 n$ is replaced by

$$c_2 n = \frac{1}{V_o s_c^{\alpha}} \quad .. (2.18)$$

where s_c = a constant

The size-mean strength relation corresponding to the above distribution function is given by

$$\bar{R} = s_o + s_c (\frac{V_o}{V})^{1/\alpha} \Gamma(1 + 1/\alpha) \quad .. (2.19)$$

where $\bar{R}$ = mean strength

Bolotin remarks that the size-mean strength relation for a wide class of materials can be given by a form such as

$$\bar{R} = \bar{R}_o [a + b (\frac{V_o}{V})^{1/\alpha}] \quad .. (2.20)$$

where $\bar{R}_o$, a, b and α are constants.

By comparing the two Eqns. 2.19, 2.20 Bolotin shows that

$$s_o = a\,\bar{R}_o \qquad \text{.. (2.21)}$$

$$s_c = b\frac{\bar{R}_o}{\Gamma(1+1/\alpha)} \qquad \text{.. (2.22)}$$

The coefficient of variation C_V of the distribution function $F_V(R_1)$ is given by

$$C_V = \frac{b\,(\frac{V_o}{V})^{1/\alpha}\;\phi_o(\alpha)}{a + b(\frac{V_o}{V})^{1/\alpha}} \qquad \text{.. (2.23)}$$

where

$$\phi_o(\alpha) = \sqrt{\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)} - 1} \qquad \text{.. (2.24)}$$

C_V = coefficient of variation

It can be noted that the results of Bolotin and Weibull are identical except for the method of approach, in that Weibull assumed the form of distribution function, while Bolotin invoked the result from extreme value statistics.

2.2.3 Reduction of Data Suitable to Weibull's Distribution Function and Applications

While there exists no formal basis for the assumed form of the distribution function of Weibull, the remarkable feature of the theory is its simplicity for application. As has been mentioned already, the basic problem of statistical strength theories is to fit experimental data into a suitable distribution function. If the form of distribution function is fixed a priori as is done by Weibull, the material parameters σ_o, σ_u and m should be adjusted to have the best statistical fit for scatter observed in experimental determination of strength of particular material. The cumulative probability of failure, according to Eqn. 2.11, is given by

$$S = 1 - \exp\left\{-V\left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m\right\} \qquad .. (2.25)$$

or $$\frac{1}{1-S} = \exp\left\{V\left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m\right\} \qquad .. (2.26)$$

or $$\log\log\frac{1}{1-S} = m\log(\sigma - \sigma_u) - m\log\sigma_o + \log V \qquad .. (2.27)$$

Eqn. 2.27 indicates that a plot of $\log\log\frac{1}{1-S}$ (as ordinate) Versus $\log(\sigma - \sigma_u)$ as abscissa should be

a straight line for any particular volume $V = V_1$, as shown in Fig. 2.2.

Further, the straight lines would be parallelly shifted with no distortion for different volumes of the same material, since σ_o and m are independent of V. In practice, the experimental data is first plotted as follows. The probability of failure S, corresponding to any stress level σ can be found from

$$S = \frac{n_\sigma}{\bar{N} + 1} \qquad .. (2.28)$$

where $\bar{N}$ = the total number of test specimens

n_σ = the number of specimens that are ruptured upto the stress σ .

While plotting the Weibull function σ_u is to be found by trial and error. First it is assumed that σ_u is zero and if the plot happens to be a straight line, that implies the assumption σ_u as zero is correct. If not a suitable value of σ_u is found by trial and error till the plot log log $\frac{1}{1-S}$ versus log $(\sigma - \sigma_u)$ is reasonably a straight line. The slope of the straight line formed earlier gives m, while the intercept on the abscissa gives σ_o. Weibull applied his theory to a number of cases viz, strength of glass rods in tension, bending strength of porcelain,

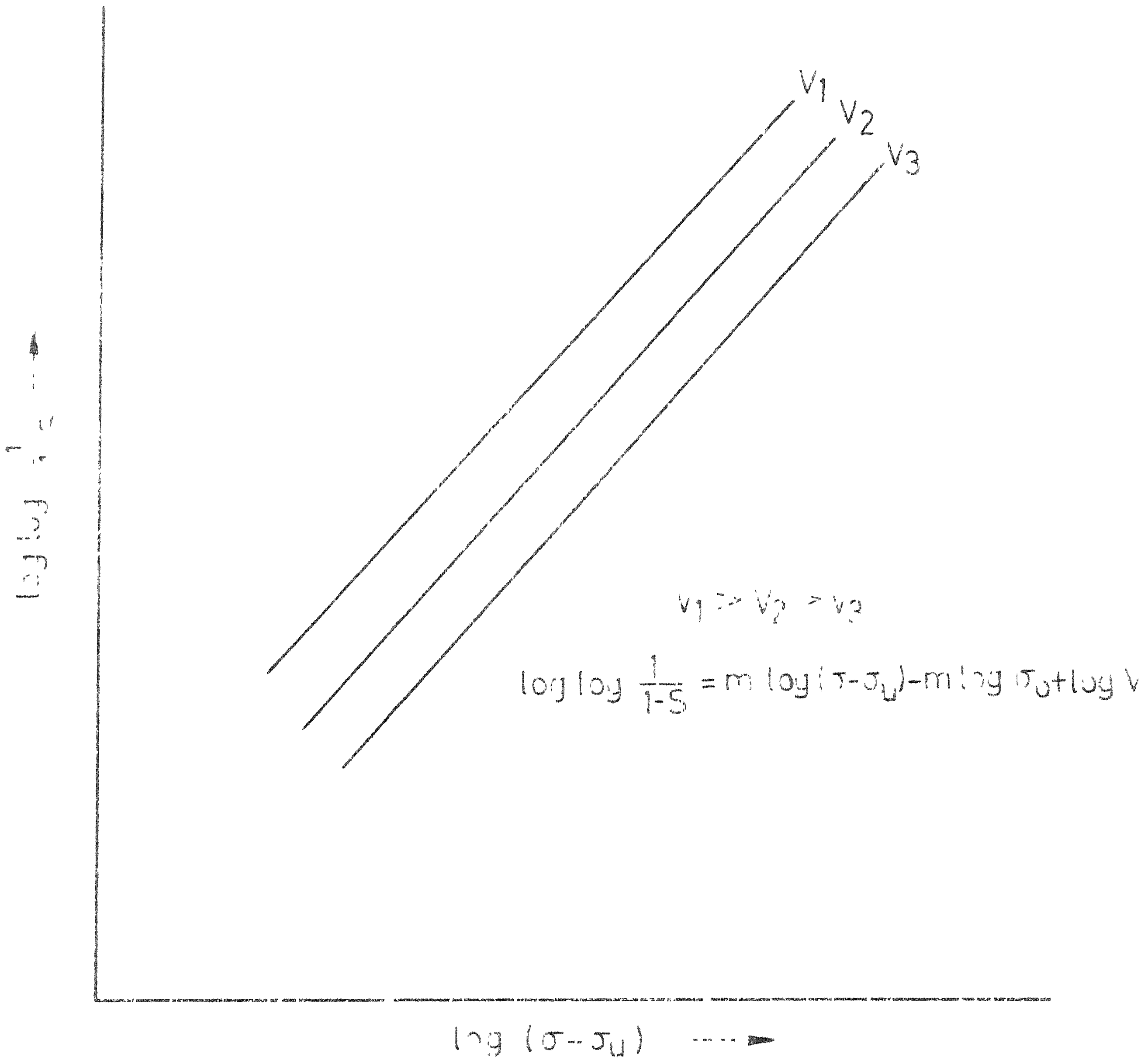


FIG. 2·2 WEIBULL PLOTS AT VARIOUS SPECIMEN SIZES
(Schematic)

tensile strength of portland cement, fiber strength of Indian cotton, breaking strength of cotton fabric strips, tensile strength of wood, stearic acid and plaster of paris, malleable iron castings etc. In all the various cases studied Weibull verified the applicability of the linear fit of the distribution function with test data satisfactorily. A complete exposition of the various methods of fitting the experimental data suitable to Weibull's distribution and the related accuracies of the methods are given by Heavens and Murgtroyd (73). It is concluded by Heavens and Murgtroyd that information from small sets of data to a three parameter Weibull distribution is not fully reliable; maximum likelihood estimates are found to be more accurate than linearization and curve fitting. This study is of practical importance and is in agreement with the findings of Fruedenthal (14).

2.3 FURTHER APPLICATIONS OF WEIBULL'S THEORY

Prompted by the work of Weibull several investigations like Davidenkov et.al. (33), Weil and Daniels (74), Greene (46), Weil et.al. (47), Durelli and Parks (75), and Evans and Pomeroy (53) examined the applicability of Weibull's distribution function to several problems of static strength of materials. Davidenkov et.al.

studied the size effect on brittle fracture of steel in the light of Weibull's theory and found the theory to be adequate in interpreting the size effect on mean strength and scatter. In addition the theory was also found to correlate the direct tensile and bending stresses at failure in that they are related by

$$\frac{\bar{\sigma}_b}{\bar{\sigma}_t} = (2m + 2)^{1/m}$$

where $\bar{\sigma}_b$ = mean failure stress in bending

$\bar{\sigma}_t$ = mean failure stress in tension

m = the material parameter

Weil and Daniels (74) derived the theoretical expressions for fracture probabilities in nonuniformly stressed specimens, such as beams under symmetrical four point loading, beams under centre point loading and beams under pure bending. The fracture probabilities in the case of pure bending derived by the authors are verified by fitting by least squares technique the analytical distribution function to their test data on Columbia Resin specimens. Greene (46) examined a possible generalization of Weibull's distribution function in the following form

$$S = 1 - \exp\left\{-V\left(\sum_{k=1}^{N} c_k \sigma^k\right)\right\} \qquad .. (2.29)$$

where c_k are material constants.

The distribution function with

$$\Sigma \; c_k \; \sigma^k = 2.9 \times 10^{-9} \; \sigma^2 + 7.15 \times 10^{-21} \; \sigma^8 \quad .. \; (2.30)$$

was found to fit well with test data on Pyrex Brand chemical glass specimens in bending, with one loading edge. As pointed by Greene, the expression relating mean strength to specimen size is extremely complicated even for the simple case of the series $\Sigma \; c_k \; \sigma^k$ where only two terms are present. Greene further verified that tests on lime glass in bending with two loading edges are adequately characterized by Weibull's distribution function.

Weil et.al (47) studied the applicability of the Weibull's theory to tests on Alumina. The applicability of the theory was satisfactory. The special feature of this particular study is the effect of parameters like surface treatment and thermal history of the material on material parameters σ_u and m has been studied. It was noticed by the authors that grinding improves the value of m, leaving the value of σ_u unaltered. Durelli and Parks (75) verified the satisfactory applicability of the linear plot of log (size) versus log (mean fracture stress), to the tests on Columbia resin

and plexiglass in tension and bending. Richards (34,35) applied Weibull's theory to characterize the scatter in the upper yield of mild steel in tension and bending and corroborated the applicability of Weibull's linear relationship between logarithm of stress at upper yield to logarithm of specimen volume . Skinner (44) examined the applicability of both the theories of Weibull (48) and Frenkel and Kontorova (76) to study the scatter in testing of Anhydrite. Even though both the theories are equally applicable with in the range of sizes of specimens tested, the author remarked that Weibull's theory is more suitable since the theory of Frenkel and Kontorova (76) is inapplicable at large sizes. This aspect will be further discussed in Sections 2.4 and 2.8 of the thesis.

The works of Evans and Pomeroy(53) and Salmassy et.al (49,50) on the application of Weibull's theory to coals tested in compression and poreclain respectively are of special relevance to the thesis. The Weibull's plots as given in the works of these authors are reproduced in Figs. 2.3 and 2.4; it could be seen from the Weibull's plots of cube tests on coal corresponding to various specimen sizes, that the plots are not parallel contradicting the assumptions of Weibull's theory. Also as remarked by Fruedenthal (14) a Weibull's

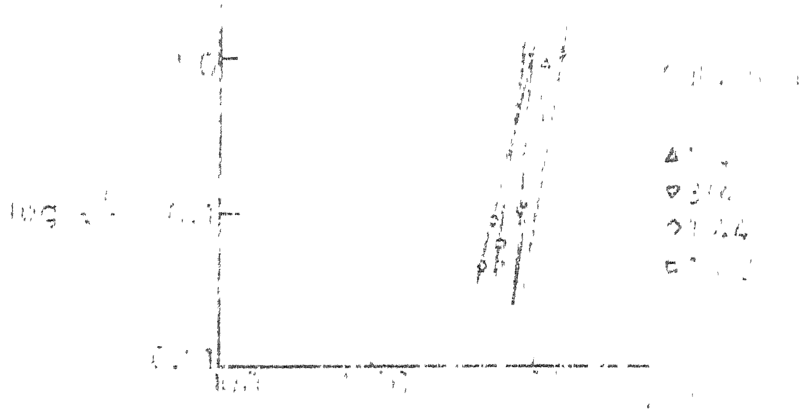


FIG. 2.3 WEIBULL PLOT OF COAL IN COMPRESSION
(DEEP DUFFERIN) (53)

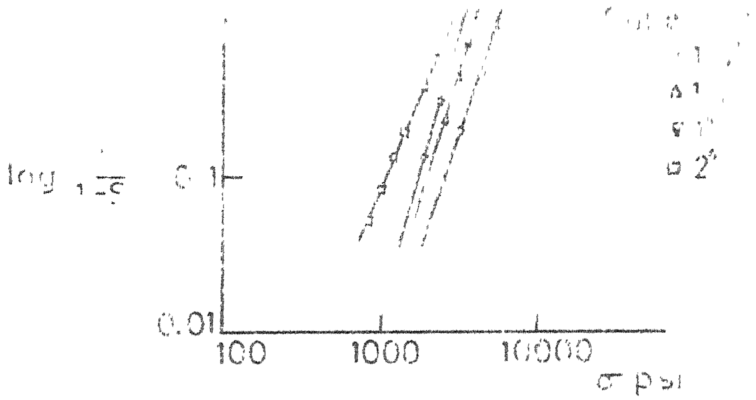


FIG. 2.4 WEIBULL PLOT OF COAL IN COMPRESSION
(BARNSLEY HARDS) (53)

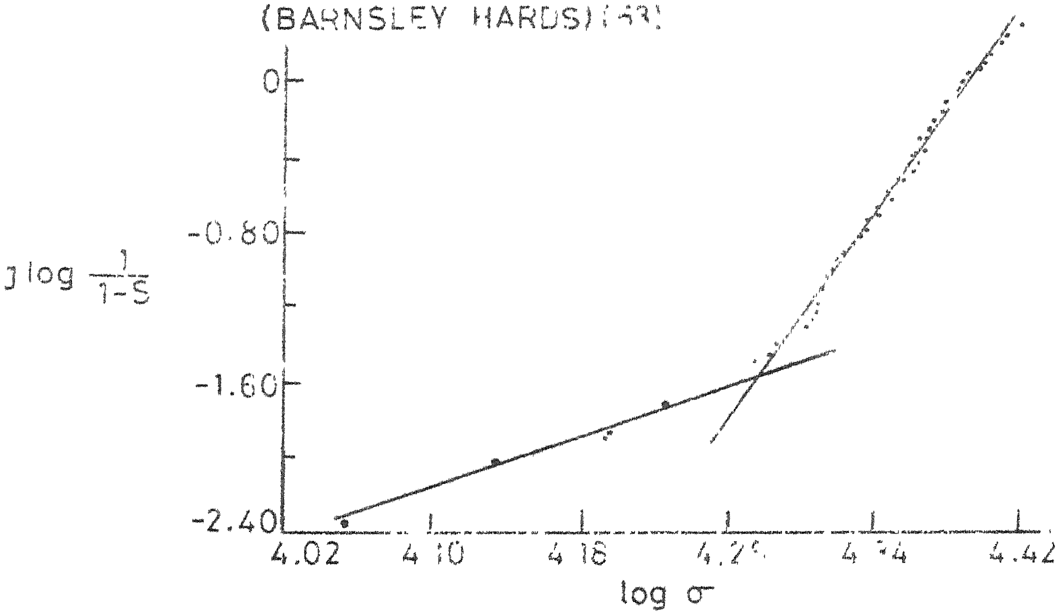


FIG. 2.5 WEIBULL PLOT OF PORCELAIN IN BENDING (14,50,59)

plot of the study by Salmassy et.al (49,50) on porcelain needs two different straight lines to characterize the strength distribution (refer to Fig. 2.5). The violation of the Weibull's theory could be due to a change in fracture mechanism at different specimen sizes, other than that assumed in the basic development. This aspect will be further discussed in Section 2.8.

2.4 THEORY OF FRENKEL AND KONTOROVA

Using the weakest link concept Frenkel and Kontorova (76) investigated the scatter in strength of materials. The authors consider that the flaws are randomly distributed in the material. Further, assuming that the flaw strengths x, are considered to be statistically normally distributed, so that

$$\Phi_o(x) = \frac{1}{\sqrt{2\pi}\; s} \exp \left\{ - \frac{(x-\mu)^2}{2s^2} \right\} \quad .. \ (2.31)$$

where μ = mean flaw strength

s = standard deviation of flaw strength

and $\Phi_o(x)$ = probability density function of flaw strength

Frenkel and Kontorova showed that the modal strength σ_m of a specimen of size V containing n flaws per unit

volume is approximately given by

$$\sigma_m \simeq \mu - s \left(2 \log nV - 2 \log 2 \sqrt{\pi}\right)^{1/2} \quad .. (2.32)$$

where σ_m = modal strength

A more accurate formula for the modal strength σ_m can be derived (77, 78), using the results of Fisher and Tippet (79) and is given by

$$\sigma_m = \mu - s \left(2 \log nV\right)^{1/2} + s \frac{\log \log nV + \log 4\pi}{2 \left(2 \log nV\right)^{1/2}} \quad .. (2.33)$$

An analytical procedure of deriving the Eqn. 2.32 is discussed by Evans and Pomeroy (53).

Apparently the Eqn. 2.32 of Frenkel and Kontorova (76) relating the specimen size and modal strength becomes invalid for very large or small values of V. For very large values of V the modal strength becomes negative, while for small values of V, due to the presence of log nV, the modal strength becomes imaginary. The reasons for the above discrepancies are that while deriving the result, the authors assume that n is very large, and the assumed normal distribution for flaw strengths associates a definite probability for negative flaw strengths.

2.5 THEORY OF FISHER AND HOLLOMON

The statistical theory of fracture due to Fisher and Hollomon (80) is essentially based on the weakest link concept and considers the random flaw occurance in the material. The theory relates the strength distribution to flaw density and consequently to the size. The basic principle is that critical flaw in a population of flaws determines the fracture stress. Instead of directly building up a strength distribution function, Fisher and Hollomon start with an assumed distribution of flaw sizes given by the density function.

$$P\left(\frac{c}{h_1}\right) = e^{-c/h_1} \qquad .. (2.34)$$

where c = the crack size

h_1 = a constant, chosen crack length

The function assumed given by Eqn. 2.34 indicates that the probability of flaw occurance decreases with flaw size. Further, it is assumed that the material fails according to Griffith's (17) law so that

$$S_1^* = A_1 c^{-1/2} \qquad .. (2.35)$$

where S_1^* = the fracture stress

c = crack size

A_1 = a constant

Defining $$S_2 = \frac{S_1^*}{A_1 h_1^{-1/2}} \quad .. (2.36)$$

Fisher and Hollomon arrive at the flaw strength density function given by

$$P(S_2) = \frac{2}{S_2^3} e^{-1/2 S_2^2} \quad .. (2.37)$$

In order to take the random flaw orientation and number into effect, the authors consider that a specimen containing n flaws will fail when the normal stress on any one flaw reaches the critical value. The problem is studied by this concept for the case of a state of stress given by

$$S_Z > 0 \; ; \; S_X = S_Y = \alpha S_Z \; ; \; \alpha \leq 1 \quad .. (2.38)$$

where X,Y,Z = there orthogonal directions

S_X, S_Y, S_Z = the stresses in the three directions.

The state of stress reduces to pure tension if $\alpha = 0$.
Using the following notation

$$S_r = \frac{S_n^*}{S_Z} \; ; \; \beta = \frac{S_Z}{A h^{-1/2}} \quad .. (2.39)$$

$$\pi = \int_{\alpha}^{\infty} P(S_r)\, dS_r - \int_{\alpha}^{1} P(S_r) \left\{ 1 - \sqrt{\frac{S_r - \alpha}{1-\alpha}} \right\} dS_r. \quad .. (2.40)$$

$$P(S_r) = \frac{2}{\beta^2 S_r^{\;3}}\, e^{-\frac{1}{\beta^2} S_r^{\;2}} \quad .. (2.41)$$

where S_n^* = the fracture stress of a specimen containing n flaws the authors show that

$$P_{n,\alpha}^{\;*}(\beta)\, d\beta = n\, \pi^{n-1} \left| \frac{\partial \pi}{\partial \beta} \right| d\beta \quad .. (2.42)$$

where $P_{n,\alpha}^{\;*}(\beta)$ = the probability that a specimen containing n flaws has a strength lying between β and $\beta + d\beta$.

The corresponding relationship for a specimen under pure tension (α=0) is

$$P_{n,0}^{\;*}(\beta) = n \left\{ 1 - \int_{0}^{1} P(S_r)\; S_r\, dS_r \right\}^{n-1}$$

$$\left| \frac{\partial}{\partial \beta} \int_{0}^{1} \{P(S_r) - \sqrt{S_r}\}\, dS_r \right| \quad .. (2.43)$$

For a given value of n , the above Eqn. 2.43 is to be evaluated numerically to evaluate $P^*_{n,0}(\beta)$. A plot of $P_{n,0}(\beta)$ versus β shows that the dispersion of strength

and mean strength decreases with increasing n. Assuming surface flaws in glass occur with a density of 10^3 per sq.cm., the authors show good agreement between mean strength and specimen size.

While the theory of Fisher and Hollomon is conceptually elegant except for the a priori assumption of the distribution of flaw sizes, an evaluation of the fracture probabilities numerically appears to be tedious. Also, a correlation of specimen size with total number of flaws is not always easily possible especially when the flaws inside the material are effective in the fracture process as in the case of metals, as remarked by the authors themselves.

2.6 THEORY OF KASE

Kase (81)in his study of the distribution of the tensile strength of rubber, considered that, of the very large number of flaws distributed throughout the material, the most critical one determines the fracture stress. The author considers an equation of the form

$$S^* = S_o^* (1 - \alpha^* x_1) \quad .. (2.44)$$

where S^* = strength of the material in the presence of a flaw

S_o^* = strength of the material in the absence of a flaw

α^* = a constant

x_1 = flaw size

With the above Eqn. 2.44, the problem of the study of the distribution of strength S^* reduces to that of the distribution of the flaw size x_1. Kase assumes that the flaws are distributed exponentially, i.e.

$$f\,(x_1) = \lambda\, e^{-\lambda x_1} \qquad .. \; (2.45)$$

where $f(x_1)$ = the probability density function of flaw size, x_1

λ = a constant.

Corresponding to this density function the most probable maximum flaw size x^* in a specimen of volume V is shown to be

$$x^* = \frac{\log Vn}{\lambda}$$

where n = the average number of flaws per unit volume.

Consequently the modal strength of a specimen S^{**}, having a volume V is given by

$$S^{**} = S_o^* \left(1 - \alpha^* \log \frac{Vn}{\lambda}\right) \qquad .. \; (2.46)$$

Assuming that the number of flaws n is very large, Kase

further shows that the strength distribution is given by

$$F(\eta) = \exp \{- \exp (-\eta)\} \qquad .. (2.47)$$

where $$\eta = \frac{\lambda}{S_o^* \alpha^*} (S^* - S^{**}) \qquad .. (2.48)$$

$F(\eta)$ = strength distribution function,

So that, knowing $\frac{\lambda}{S_o^* \alpha^*}$ and S^{**} one can readily get the distribution of strength S^*. Experimental investigations yield the values of mean strength $\bar{\sigma}$ and standard deviation S_d. Corresponding to the above distribution function it has been shown that

$$S^{**} - \frac{S_o^* \alpha^*}{\lambda} \cdot 0.1577 = \bar{\sigma} \qquad .. (2.49)$$

$$\frac{S_o^* \alpha^*}{\lambda} \frac{\pi}{\sqrt{6}} = S_d \qquad .. (2.50)$$

With a knowledge of $\bar{\sigma}$ and S_d, the function $F(\eta)$ in Eqn. 2.47 can be characterized. Kase applied his theory to verify the scatter in his tension test results on rubber and found the predicted form of $F(\eta)$ to be agreeable with the experimental one.

2.7 OTHER PHENOMENOLOGICAL THEORIES

The statistical theory of strength due to Valkov (82) is a development of the basic concepts of continuum

mechanics to random micro-media or a microscopically inhomogeneous medium. In contrast to the theories reviewed earlier, the theory of Valkov is devoted to the determination of stress fields in a random medium and therefore is a study of statistical continuum theory analogous to the work of Beran (102). Brady (83) advanced a failure criterion for rock like materials in which microcracks develop and propagate under increased loading. Considering the microcrack density at various stages of loading upto failure and the rupture of material according to Mohr - Coulomb criterion, Brady advanced a failure criterion that, fracture occurs when the total volumetric strain due to microcracks attains a critical value. The theory is shown to be in qualitative agreement with various aspects of mechanical behaviour of rock like materials and is used to explain the nonlinear stress strain curve as affected by microcracking.

It may be noted that the two theories mentioned in this section, eventhough are termed as statistical strength theories, treat altogether a different problem from that treated by Weibull and others. The term "statistical aspects of strength and fracture" is used for the study of creep - rupture of concrete as a

stochastic process by Hori (129). The aforesaid theories do not deal with strength specifically as related to size although the mechanical behaviour is explained on the basis of random internal structure of the material.

2.8 DISCUSSION

The application of weakest link concept in the study of statistical strength theories of materials is popular because of the simplicity in mathematical formulation of the problem. However, the implications of the weakest link concept are vulnerable to some criticism. In the study of a continuous material in the presence of randomly distributed flaws, the weakest link concept centres attention only on the most critical or potential flaw and the interaction between the various flaws is neglected. Qualitatively, the essence of the concept is that larger the specimen size, greater is the probability of a critical flaw occuring. As such, a majority of the studies like that of Weibull (48) and Frenkel and Kontorova (76) have been shown by Epstein (78,84), Saibel (85) and Romauldi (86) to belong to a class of problems in extreme value statistics, in which the distribution of the smallest value in a sample taken from a population is studied. Apart from the statistical distribution of flaws in the material, the various theories

based on weakest link concept do not take care of any of the aspects of actual mechanisms of fracture in all types of materials. Table 2.1 gives a comparative tabulation of the assumptions in the mathematical formulation and the final results of the various theories discussed in the preceding sections.

2.8.1 Comparisons and Limitations of the above Theories

A general limitation of the various theories based on weakest link concept reviewed above is an adhoc and a priori assumption of the form of strength distribution function, flaw strength distribution or flaw size distribution. Eventhough an assumption of some kind is justifyable on the grounds of simplicity, the real problem of identifying a distribution to fit the scatter in materials strength results obtained by actual tests is circumvented. Consequently, the true fracture mechanisms are wiped out and therefore it should at best be considered as a hypothesis or an assumption which should be checked while trying to apply to a particular material. By such assumptions, the characteristic features of the scatter of the material are a priori fixed by the assumption made.

Inspite of its remarkable simplicity the distribution function assumed by Weibull (48) automatically

TABLE 2.1: VARIOUS STATISTICAL STRENGTH THEORIES: TABULATED COMPARISON

S.No.	Theory	Assumption	Final Result
1	Weibull	Strength Distribution Function: $S = 1 - \exp \{-V \frac{\sigma - \sigma_u}{\sigma_o}^m\}$	Size-Mean Strength Relation: $\bar{\sigma} = \sigma_u + \frac{\sigma_o}{V^{1/m}} \Gamma(1+1/m)$
2	Frenkel and Kontorova	Flaw Strength Distribution: $\Phi_o(x) = \frac{1}{\sqrt{2\pi}\,.s} \exp \{-\frac{(x-\mu)^2}{2s^2}\}$	Size-Modal Strength Relation: $\sigma_m \approx \mu - s\{2\log nV - 2\log 2\sqrt{\pi}\}^{1/2}$
3	Fisher and Hollomon	Crack Length Distribution: $P(c/h_1) = \exp(-c/h_1)$	Density Function for Strength: $P^*_{n,o}(\beta) = n\{1 - \int_o^1 P(S_r)\sqrt{S_r}\, dS_r\}^{n+1}$ $\vert \frac{\partial}{\partial\beta} \int_o^1 \{F(S_r - \sqrt{S_r})\}\, dS_r \vert$
4	Kase	i) Strength-Crack Size Relation: $S^* = S^*_o(1 - \alpha^* x_1)$ ii) Crack Size x_1 is exponentially distributed	Strength Distribution Function: $F(\eta) = \exp\{-\exp(-\eta)\}$ $\eta = \frac{\lambda}{S^*_o\alpha^*}(S^* - S^{**})$ $S^{**} = S^*_o(1 - \alpha^* \log \frac{Vn}{\lambda})$

fixes the form of size-mean strength relation. It can be easily observed that the actual mechanism at fracture can be different at different specimen sizes, like for example, depending on the energy content for driving a potential crack. If the specimen is large, the total energy of the material under load is higher and consequently the crack driving force is more, than that of a smaller specimen. The larger specimens with a potential crack are likely to fail in a brittle manner as compared to smaller ones. Consequently, it may not be always possible to characterize the size mean strength relation by a simple a priori fixed form.

Frenkel and Kontorova (76), instead of assuming a distribution function directly assume that the flaw strengths in the material are distributed normally and that the density of flaws is very large. As a consequence the size-modal strength relation obtained by this theory is inapplicable at very large or small volumes. Physically it can be seen that at very large sizes the strength should approach a finite value and also at smaller volumes the theoretical strength of the material in the absence of any flaws.

Fisher and Hollomon (80) assume flaw length distribution in the material and relate scatter in

strength with the density of flaws. Eventhough the results of the theory are not available in closed form, it can be expected that the assumed form of flaw distribution influences the form of the strength distribution as well as size-mean strength relationship. The theory of Kase (81), assumes the nature of flaw distribution as well as the relation between strength and flaw size. The applicability of the relationship between modal strength and specimen size of Kase is limited at very large volumes.

Of the many theories refered to above, that of Weibull (48) is applied to a wide variety of problems. However, as could be seen the applicability of the theory is limited in the case of some materials which will be discussed further in the thesis.

2.8.2 Scope for Alternative Approach

Inspite of the assumptions implicit in the weakest link concept viz, that only the potential flaw determines material strength and flaw interaction effects are neglected, the application of the concept is still welcome because of the mathematical simplicity in studying the problem of characterization of scatter in materials testing. However, as could be noticed from the theories reviewed earlier any a priori assumption

of the form of strength distribution function or flaw size distribution essentially limits the scope, in that the form of size mean strength relation is fixed. As could be seen from the studies of Evans and Pomeroy (53) on coal in compression, the Weibull plots at various specimen sizes, Figs. 2.3 and 2.4, eventhough straight lines, are not parallel. This implies that the parameter 'm' in the Weibull's distribution that determines the slope of the straight line in the Weibull's plot is a function of specimen volume. In effect this means that the assumed form of distribution function is incorrect. Similarly the results of Salmassy et.al (49,50) on porcelain indicate the need for two straight lines in Weibull's plot, Fig.2.5, to characterize the scatter as also remarked by Fruedenthal (14). These remarks show the limitations of applicability of the above mentioned formulations for all solid materials equally well, because of the inherent assumptions. A possible approach to by pass these problems is to seek a methodology by which a strength distribution function can be arrived at formally, suitable to a particular material as given by its size-mean strength relation. In this manner, the true material characteristics can be reflected in the formulation. A formulation of this problem and its solution in special case is discussed in the following chapters.

CHAPTER THREE

PROPOSED FORMULATION OF THE PROBLEM OF SIZE EFFECT ON STRENGTH OF MATERIALS

3.1 BASIC MOTIVATION FOR THE PRESENT FORMULATION

From the review of various statistical strength theories in the earlier chapter, one can notice that the weakest link concept, even though an idealization of material's mechanism of fracture, does provide a basis for mathematical formulation and application. Furthermore, the various theories discussed in Chapter 2 start with an a priori assumption of the form of strength distribution function. As a consequence of such assumption, the form of size-mean strength relation or size-modal strength relation is automatically fixed. The accuracy in the application of any particular theory lies in the problem of how best the material under consideration can be characterized by the size-mean strength relation or size-modal strength relation corresponding to the applied theory. However, as mentioned in Section 2.8.2, application of Weibull's hypothesis to data available on coal and porcelain showed the limitations of the basic formulation in taking care of behaviour of these materials. A plausible way of finding a more suitable distribution function

reflecting observed material behaviour, could be made by constructing the same from first principles taking the size-mean strength relation as the characteristic of the material and getting the distribution function corresponding to the material rather than assuming the form of the distribution function a priori. In the present Chapter, possibilities of this approach are examined in detail.

The motivation for the present work is that of Tsai and Kolsky (87) in which they examined the apparent fracture strength of glass plates as dependent on the diameter of the indentor. The authors constructed a theoretical strength distribution function corresponding to the mean fracture strength - indentor radius relation as given by experimental investigations. The authors concluded that scatter prediction corresponding to large indentor strengths is satisfactory by the statistical approach.

In this chapter, the close analogy of the problem of size effect on strength and that of indentor size on fracture strength of glass as attempted by Tsai and Kolsky (87) is used for formulation of size effects on strength of materials which is an alternative approach to presently available theories (as pointed out

in Section 2.8).

3.1.1 Specification of the Problem

In the following, the problem of constructing the strength distribution function that corresponds to a given form of size-mean strength relation (obtained experimentally) is attempted. The distribution function is required to reveal the general features of mechanical behaviour from typical tests on materials for strength with size as a parameter, viz, the decrease in scatter with increase in specimen size. The method of approach and solution is developed for this problem in a manner similar to that of Tsai and Kolsky (87) in their investigation of tensile strength of glass as dependent on indentor diameter.

3.2 ASSUMPTIONS MADE IN THE FORMULATION

The basic assumptions of the nature of failure of material are the same as those of Weibull, viz, that the material obeys weakest link concept in that the fracture of any typical element is the same as that of the total material; the flaws initiating rupture are randomly distributed throughout the material and the interaction between various elements with flaws is neglected. The most potential flaw that forms the weakest link determines the rupture of material.

3.3 THEORETICAL MODEL OF THE PROBLEM OF SIZE EFFECTS ON STRENGTH

As has been discussed in Section 2.1 of the thesis, a general feature of the tests on strength of materials is the decrease in mean strength and scatter with increasing size. The size could correspond to volume, area or length depending on the nature of material and their testing. In the following presentation, the volume of the material is considered to represent size and a change to area or length, if need be, depending on the situation does not pose any problem.

Considering the strength of a specimen of size V, let the total volume be divided into several infinitismal elements of size ΔV. Let the probability, that a potential flaw exists in an elemental volume ΔV, and the flaw is of such size that fracture occurs when the applied stress is between σ and $\sigma + d\sigma$, be defined as $\{\phi(\sigma)\, d\sigma\}\ \Delta V$. It may be noted that this definition is analogous to that of the occurance of an event in an interval Δt in a Poisson process of parameter λ which is given by λdt. In the present problem, it so happens that the corresponding parameter is dependent on the applied stress σ and is given by $\phi(\sigma)\, d\sigma$. Having defined $\{\phi(\sigma)\, d\sigma\}\ \Delta V$ as above, it can be written that the probability $P(\sigma)$ that

fracture occurs in the element under consideration for any stress level lying between 0 and σ be given by

$$P(\sigma) = \{ \int_0^\sigma \phi(\sigma)\, d\sigma \} \times \Delta V \qquad .. (3.1)$$

In many materials the size-mean strength relation converges asymptotically to a value σ_L , the lowest mean strength and this can be incorporated in the analysis by specifying

$$\phi(\sigma) = 0 \qquad \text{for} \quad \sigma < \sigma_L \qquad .. (3.2)$$

Using Eqns. 3.1 and 3.2 the probability $P^*(\sigma)$ that fracture cannot occur in the elemental volume ΔV for a stress level lying between 0 and σ is given by

$$P^*(\sigma) = 1 - \{ \int_{\sigma_L}^\sigma \phi(\sigma)\, d\sigma \} \quad \Delta V = 1 - G(\sigma)\, \Delta V$$

$$\text{where } G(\sigma) = \int_{\sigma_L}^\sigma \phi(\sigma)\, d\sigma \qquad .. (3.3)$$

If the total volume V of the specimen is considered to be divided into several infinitismal volumes ΔV_i $\{ i = 1,2 \ldots \infty \}$, the probability that failure cannot occur anywhere in the total specimen $\Phi(\sigma)$ is given by

$$\Phi(\sigma) = \pi \{ 1 - G(\sigma)\, \Delta V_i \} \ ; \ i=1,2,3 \ldots \qquad .. (3.4)$$

It may be noticed that in writing Eqn. 3.4, the weakest link concept is implicitly used in the sense that the failure of total material is considered to be same as that of any one of the elements ΔV_i failing.

The infinite product in Eqn. 3.4 may be simplified using the following inequalities. For any x_i, so that $0 < x_i < 1$ one can write

$$e^{-x_i} > (1-x_i) > e^{-x_i} - 1/2 \; x_i^2 \qquad .. (3.5)$$

or $$\pi(1-x_i) < \pi\, e^{-x_i} = e^{-\Sigma\, x_i} \qquad .. \; .$$

and $$\pi(1-x_i) > e^{-\Sigma\, x_i}\, \pi(1-1/2\; x_i^2\; e^{x_i}) \qquad .. (3.6)$$

In the limit, as $x_i \to 0$, the above inequalities indicate that

$$\pi\,(1-x_i) \to e^{-\Sigma\, x_i} \qquad .. (3.7)$$

Identifying x_i as $G(\sigma)\;\Delta V_i$ in the Eqn. 3.4, one can write

$$\pi\,(1-G(\sigma)\,\Delta V_i) = e^{-G(\sigma)\,\Sigma\,\Delta V_i}$$

as $\Delta V_i \to 0$;

$$\pi\,(1-G(\sigma)\,\Delta V_i) = e^{-G(\sigma)\int_V dV} = e^{-G(\sigma)\,V}$$

$$= e^{-V\int_{\sigma_L}^{\sigma} \phi(\sigma)\,d\sigma} \text{ from Eqn.3.3} \qquad .. (3.8)$$

Since $\Phi(\sigma)$ is the probability that failure does not occur anywhere in the specimen of volume V, the probability $\Phi^*(\sigma)$ that fracture occurs for any stress lying between 0 and σ, is given by

$$\begin{aligned}
\Phi^*(\sigma) &= 1 - \Phi(\sigma) = 1 - e^{-V\,G(\sigma)} \\
&= 1 - e^{-V\int_{\sigma_L}^{\sigma} \phi(\sigma)\,d\sigma} && \text{for } \sigma > \sigma_L \\
&= 0 && \text{for } \sigma \leq \sigma_L
\end{aligned} \qquad .. (3.9)$$

The density function $f(\sigma)$, corresponding to $\Phi^*(\sigma)$, namely the probability that fracture occurs for an applied stress level lying between σ and $\sigma + d\sigma$ is given by

$$f(\sigma) = \frac{d\,\Phi^*(\sigma)}{d\sigma} \qquad .. (3.10)$$

Also, if $\bar{\sigma}$ is the mean fracture stress, one has

$$\bar{\sigma} = \frac{\int_0^\infty \sigma\, f(\sigma)\, d\sigma}{\int_0^\infty f(\sigma)\, d\sigma} = \int_0^\infty \sigma f(\sigma)\, d\sigma = \int_{\sigma_L}^\infty \sigma f(\sigma)\, d\sigma \qquad .. (3.11)$$

Since $f(\sigma) = 0$ for $\sigma \leq \sigma_L$

Integrating Eqn. 3.11 by parts and evaluating the limits

$$\bar{\sigma} = \int_{\sigma_L}^{\infty} \sigma \frac{d\,\Phi^*(\sigma)}{d\sigma}\, d\sigma = \left[\sigma \quad \Phi^* (\sigma) - \int \Phi^*(\sigma)\, d\sigma \right]_{\sigma_L}^{\infty}$$

$$= \{\sigma\Phi^* (\sigma) \}_{\sigma_L}^{\infty} - \int_{\sigma_L}^{\infty} \{ 1 - e^{-V \int_{\sigma_L}^{\sigma} \phi(\xi)\, d\xi} \}\, d\sigma$$

or

$$\bar{\sigma} = \sigma_L + \int_{\sigma_L}^{\infty} \{ e^{-V \int_{\sigma_L}^{\sigma} \phi(\xi)\, d\xi} \}\, d\sigma \qquad .. (3.12)$$

where ξ is a dummy variable of integration.

The above Eqn. 3.12 is the relationship that relates specimen size V to the mean strength $\bar{\sigma}$. If $\phi(\xi)$ or $\phi(\sigma)$ is evaluated corresponding to a given size-mean strength relation, one can find $f(\sigma)$ which is the density function denoting the probability of failure, of specimen for a stress lying between σ and $\sigma + d\sigma$ and $\Phi^*(\sigma)$ which is cumulative probability of failure using Eqns. 3.9 and 3.10. A method of solution of Eqn. 3.12 will be discussed in detail in the next Chapter. It may be noted that Eqn. 3.12 is a non-linear integral equation.

3.4 DISCUSSION AND COMPARISON WITH THE EXISTING THEORIES

In contrast with the methods of any of the statistical strength theories (Chapter 2), the present approach seeks a method by which the strength distribution function can be constructed corresponding to a given form of size-mean strength relationship, with no a priori assumption of the form of strength distribution or flaw distribution function. The present method in effect is converse in its method of approach to existing ones, in that the distribution function is obtained subsequent to a given size-mean strength relationship while in all the theories of Weibull and others, some assumption of strength distribution is made (Chapter 2) and size-mean strength or size - modal strength relation is obtained which is fitted with the experimental data.

While the ease and simplicity with which the theories of Weibull and others can be applied to some materials, are undisputable, it is imperative that alternative methods of approach should be available to treat cases of materials that do not suit the available theories like for example the coal cubes and porcelain results discussed in Section 2.8.1, wherein the Weibull plots corresponding to various sizes are not parallel for coal cubes and two straight lines are required to fit

data on porcelain. While a possible approach is to try alternate forms of distribution functions, it may be relatively easier and elegent to try alternate forms of size-mean strength relationships corresponding to the actual experimental data and the problem formulated is aimed at devising a general method by which distribution functions can be obtained, given the size-mean strength relationship.

$$\bar{\sigma} = \sigma_L \quad \{1 + \frac{K}{V^n}\} \qquad .. (4.3)$$

where σ_L is the mean lowest strength; K and n are material parameters.

Eqn. 4.3 chosen above indicates that as V increases $\bar{\sigma}$ decreases and tends to σ_L asymptotically as V tends to be very large.

Substituting Eqn. 4.3 in 4.1

$$\frac{\sigma_L K}{V^n} = \int_{\sigma_L}^{\infty} \{e^{-V \int_{\sigma_L}^{\sigma} \phi(\xi)\, d\xi}\} \, d\sigma \qquad .. (4.4)$$

To simplify the above Eqn. 4.4, let the following notation be used

$$\int_{\sigma_L}^{u} \phi(\xi)\, d\xi = \psi(u) \qquad .. (4.5)$$

so that

$$\frac{\sigma_L K}{V^n} = \int_{\sigma_L}^{\infty} e^{-V \psi(u)} \, du \qquad .. (4.6)$$

Differentiating Eqn. 4.5 under the integral sign

$$\frac{d\,\psi(u)}{du} = \psi'(u) = \phi(u) \qquad .. (4.7)$$

so that, knowing $\psi(u)$, $\phi(u)$ can be evaluated by differentiating $\psi(u)$.

To evaluate $\psi(u)$, let

$$\psi(u) = t \qquad .. (4.8)$$

and

$$u = u(t) \qquad .. (4.9)$$

so that

$$du = u' \, dt \qquad .. (4.10)$$

Assume that as $u \to \sigma_L$, $t \to \alpha_o$ and as $u \to \infty$, $t \to \infty$ where α_o is any constant greater than or equal to zero.

With this notation, Eqn. 4.6 takes the form

$$\frac{\sigma_L K}{V^n} = \int_{\alpha_o}^{\infty} e^{-Vt} u' \, dt \qquad .. (4.11)$$

or equivalently, by dividing either side by V

$$\int_{\alpha_o}^{\infty} \frac{e^{-Vt}}{V} u' \, dt = \sigma_L K V^{-(n+1)} \qquad .. (4.12)$$

Taking the Inverse Laplace Transform of Eqn. 4.12 with respect to V one obtains

$$\int_{\alpha_o}^{\infty} u' H(s-t) \, dt = \frac{\sigma_L K s^n}{\Gamma(n+1)} \qquad .. (4.13)$$

where $H(s-t)$ is the Heaviside unit function given by

$$\begin{aligned} H(s-t) &= 0 \qquad \text{for } s < t \\ &= 1 \qquad \text{for } s > t \end{aligned} \qquad .. (4.14)$$

s = is the parameter in the Laplace Transform

and $\Gamma(n+1)$ = gamma function of argument $(n+1)$

Using the properties of H(s-t) given by Eqn. 4.14, Eqn. 4.13 takes the form

$$\int_{\alpha_0}^{s} u' \ dt = \frac{\sigma_L K \ s^n}{\Gamma (n+1)} \quad .. (4.15)$$

or $$u(s) - u(\alpha_0) = \frac{\sigma_L K \ s^n}{\Gamma (n+1)} \quad .. (4.16)$$

Eqn. 4.16 can be rewritten as

$$u(t) = u(\alpha_0) + \frac{\sigma_L K \ t^n}{\Gamma (n+1)}$$

$$= \sigma_L \{1 + \frac{K \ t^n}{\Gamma (n+1)}\} \quad .. (4.17)$$

It may be verified from Eqn. 4.17 that the assumption made in introducing u, viz, that as $t \to \infty$, $u \to \infty$ is satisfied. From Eqn. 4.8

$$\psi (u) = t$$

Eqn. 4.17 can be rewritten as

$$u = \sigma_L \{1+ \frac{K \psi(u)^n}{\Gamma (n+1)} \} \quad .. (4.18)$$

so that

$$\psi(u) = \{\frac{\Gamma (n+1) \ (u-\sigma_L)}{K \ \sigma_L} \}^{1/n} \quad .. (4.19)$$

Since from Eqn. 4.7

$$\phi (u) = \psi' (u)$$

$\phi(u)$ can be evaluated using Eqn. 4.19 as follows

$$\phi(u) = \psi'(u) = \left\{\frac{\Gamma(n+1)}{K\,\sigma_L}\right\}^{1/n} \cdot \frac{1}{n} \cdot (u - \sigma_L)^{(1/n)-1} \quad .. (4.20)$$

From Eqn. 4.20 and the restriction on $\phi(\sigma)$ for $\sigma \leq \sigma_L$ (See Eqn. 3.2 of Section 3.3 in the previous Chapter) one has

$$\begin{aligned}
\phi(\sigma) &= \left\{\frac{\Gamma(n+1)}{K\,\sigma_L}\right\}^{1/n} \cdot \frac{1}{n} \cdot (\sigma - \sigma_L)^{(1/n)-1} && \sigma > \sigma_L \\
&= 0 && \sigma \leq \sigma_L \qquad .. (4.21)
\end{aligned}$$

Eqn. 4.21 is the solution of Eqn. 4.1 corresponding to the size-mean strength relation of Eqn. 4.3. By direct substitution it can be verified that $\phi(\sigma)$ from Eqn. 4.21 satisfies Eqn. 4.1. The expressions for $\Phi^*(\sigma)$ and $f(\sigma)$ are given below using Eqns. 4.21, 3.9 and 3.10 of the preceding chapter.

$$\begin{aligned}
\Phi^*(\sigma) &= 1 - \exp\left[-V\left\{\frac{\Gamma(n+1)}{K\,\sigma_L}\right\}^{1/n} \cdot (\sigma - \sigma_L)^{1/n}\right] && \text{for } \sigma > \sigma_L \\
&= 0 && \text{for } \sigma \leq \sigma_L \qquad .. (4.22)
\end{aligned}$$

and

$$\begin{aligned}
f(\sigma) &= \frac{V}{n}\left\{\frac{\Gamma(n+1)}{K\,\sigma_L}\right\}^{1/n} \cdot (\sigma - \sigma_L)^{1/n - 1} \times \\
&\quad \exp\left[-V\left\{\frac{\Gamma(n+1)}{K\,\sigma_L}\right\}^{1/n} \cdot (\sigma - \sigma_L)^{1/n}\right] && \text{for } \sigma > \sigma_L \\
&= 0 && \text{for } \sigma \leq \sigma_L \qquad .. (4.23)
\end{aligned}$$

Corresponding to $\phi^*(\sigma)$ and $f(\sigma)$ given by Eqns. 4.22 and 4.23 above, the standard deviation S_d is given by

$$S_d = \left\{ \int_0^\infty \sigma^2 f(\sigma)\, d\sigma - \bar{\sigma}^2 \right\}^{1/2}$$

$$= \frac{\sigma_L K}{V^n} \sqrt{\frac{\Gamma(2n+1)}{\Gamma^2(n+1)} - 1} \qquad .. \; (4.24)$$

where S_d = standard deviation

The coefficient of variation C_v is given by

$$C_v = \frac{1}{\left(1 + \frac{V^n}{K}\right)} \sqrt{\frac{\Gamma(2n+1)}{\Gamma^2(n+1)} - 1} \qquad .. \; (4.25)$$

4.1.1 <u>Specialization to Weibull's Result</u>

Weibull's strength distribution function is given by

$$\phi^*(\sigma) = 1 - e^{-V\left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m} \qquad .. \; (4.26)$$

and the size-mean strength relation and the variance S_d^2 corresponding to this distribution function are given by

$$\bar{\sigma} = \bar{\sigma}_u + \frac{\sigma_o}{V^{1/m}} \Gamma(1+1/m) \qquad .. \; (4.27)$$

$$S_d^2 = \sigma_o^2 \, V^{-2/m} \left\{ \Gamma(1+2/m) - \Gamma^2(1+1/m) \right\} \qquad (4.28)$$

The size-mean strength relation considered in the present formulation is given by

$$\bar{\sigma} = \sigma_L \left\{ 1 + \frac{K}{V^n} \right\} \qquad .. \; (4.29)$$

and the variance is given by the square of standard deviation

$$(S_d)^2 = \frac{\sigma_L^2 K^2}{V^{2n}} \left(\frac{\Gamma(2n+1)}{\Gamma^2(n+1)} - 1\right) \quad .. (4.30)$$

Comparing Eqns. 4.29 and 4.27 the following correspondence between parameters in Weibull's equations and those in the present formulation can be established

$$\sigma_L \rightarrow \sigma_u ; \quad \sigma_L K \rightarrow \sigma_o \times \Gamma(1+1/m) \text{ and } n \rightarrow 1/m \quad .. (4.31)$$

Using the relations 4.31 in Eqn. 4.30

$$(S_d)^2 = \frac{\sigma_o^2\, \Gamma^2(1+1/m)}{V^{2/m}} \left\{ \frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1 \right\}$$

$$= \sigma_o^2 \quad V^{-2/m} \left\{ \Gamma(1+2/m) - \Gamma^2(1+1/m) \right\} \quad .. (4.32)$$

which is the same as Weibull's result

4.1.2 Specialization to Bolotin's Result

In the notation of Bolotin (72) the coefficient of variation in strength as affected by specimen size is given by (Refer Section 2.2.2)

$$C_v = \frac{b\,(\frac{V_o}{V})^{1/\alpha}\ \phi(\alpha)}{a + b\,(\frac{V_o}{V})^{1/\alpha}} \quad .. (4.33)$$

$$\text{where } \phi(\alpha) = \sqrt{\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)} - 1} \quad .. (4.34)$$

The size-mean strength relation of Bolotin is of the form

$$\bar{R} = R_o \left\{ a + b\,(\frac{V_o}{V})^{1/\alpha} \right\} \quad .. (4.35)$$

where $\bar{R}$ = the mean strength

$\bar{R}_o, a, b, \alpha$ = are constants

V_o = some standard volume

The corresponding results of the present approach are

$$C_v = \frac{1}{\left(1+\frac{V^n}{K}\right)} \sqrt{\frac{\Gamma(2n+1)}{\Gamma^2(n+1)} - 1} \quad \text{.. (4.36)}$$

and $$\bar{\sigma} = \sigma_L \left\{1+\frac{K}{V^n}\right\} \quad \text{.. (4.37)}$$

Comparing Eqns. 4.35 and 4.37 the following correspondence between various constants can be noted

$$\bar{\sigma} \to \bar{R}, \quad \sigma_L \to \bar{R}_o\, a; \quad \sigma_L K \to \bar{R}_o\, b\, V_o^{\,1/\alpha}$$

and $n \to 1/\alpha$

The coefficient of variation given by Eqn. 4.36, with the above changes in notation is given by

$$C_v = \frac{1}{1+\dfrac{V^{1/\alpha}.\bar{R}_o.a}{\bar{R}_o\, b\, V_o^{\,1/\alpha}}} \times \sqrt{\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)}} \quad \text{.. (4.38)}$$

$$= \frac{b(V_o)^{1/\alpha}}{b\, V_o^{\,1/\alpha}+V^{1/\alpha}.a} \sqrt{\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)} - 1}$$

$$= \frac{b(\frac{V_o}{V})^{1/\alpha}}{a+b(\frac{V_o}{V})^{1/\alpha}} \sqrt{\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)} - 1}$$

Which is the same as that given by Eqn. 4.33.

4.2 FEASIBILITY OF CLOSED FORM SOLUTIONS

The solution of Eqn. 4.1 for $\phi(\sigma)$ corresponding to a given form of size-mean strength relationship will be in closed form whenever the inverse Laplace transform in Eqn. 4.12 as well as an explicit form of $\psi(u)$ in Eqn. 4.18 can be obtained. However the above restrictions are stringent in that both may not be obtained without resort to numerical procedures. Tsai and Kolsky (87), in their attempt to get a fracture stress distribution function (corresponding to a given indentor radius-mean fracture load relation), obtained the distribution function in closed form only for a quadratic relation between indentor radius and fracture load.

In the following, the solution of Eqn. 4.1 for $\phi(\sigma)$ corresponding to a more general form of size-mean strength relationship given by

$$\bar{\sigma} = \sigma_L \left\{ 1 + \frac{N(V)}{D(V)} \right\} \quad \ldots (4.39)$$

where $D(V)$ is a polynomial of degree n

$N(V)$ is a polynomial of degree n-1 or less

is attempted. The form of Eqn. 4.39 reflects that as V increases $\bar{\sigma}$ decreases and approaches σ_L in the limit. The following procedure is analogous to what

has been followed in the earlier section.

Substituting Eqn. 4.39 in Eqn. 4.1

$$\frac{\sigma_L N(V)}{D(V)} = \int_{\sigma_L}^{\infty} e^{\{-V \int_{\sigma_L}^{\sigma} \phi(\xi)\, d\xi\}} \, d\sigma \qquad .. (4.40)$$

or
$$\frac{\sigma_L N(V)}{D(V)} = \int_{\sigma_L}^{\infty} e^{-V \psi(u)} \, du \qquad .. (4.41)$$

and
$$\frac{\sigma_L N(V)}{VD(V)} = \int_{\alpha_o}^{\infty} \frac{e^{-Vt}}{V} u' \, dt \qquad .. (4.42)$$

where $\psi(u)$, t, and α_o are defined exactly as before.

Taking the Inverse Laplace transform of Eqn. 4.42 on either side with respect to V

$$\int_{\alpha_o}^{\infty} u' \, H(s-t)\, dt = L^{-1} \{\frac{\sigma_L N(V)}{VD(V)}\} \qquad .. (4.43)$$

or
$$\int_{\alpha}^{s} u' \, dt = L^{-1} \{\frac{\sigma_L N(V)}{VD(V)}\} \qquad .. (4.44)$$

where L^{-1} denotes Inverse Laplace transform. The quantity on the right hand side of Eqn. 4.44 can be evaluated as (88)

$$L^{-1} \{\frac{\sigma_L N(V)}{VD(V)}\} = \sigma_L \sum_{m=1}^{n+1} \frac{N(a_m)}{D'(a_m)} e^{a_m s} \qquad .. (4.45)$$

where $a_1, a_2, \ldots a_{n+1}$ are the roots of the equation

$$\bar{D}(V) = V\,D(V) = 0$$

and $\bar{D}'$ denotes the derivative of the polinomial $\bar{D}$.

From Eqns. 4.44 and 4.45

$$u(s) - u(\alpha_0) = u(s) - \sigma_L = \sigma_L \sum_{m=1}^{n+1} \frac{N(a_m)}{\bar{D}'(a_m)} e^{a_m s} \quad \text{.. (4.46)}$$

or

$$u(s) = \sigma_L \left\{ 1 + \sum_{m=1}^{n+1} \frac{N(a_m)}{\bar{D}'(a_m)} e^{a_m s} \right\} \quad \text{.. (4.47)}$$

or

$$\left\{ \frac{u - \sigma_L}{\sigma_L} \right\} = \sum_{m=1}^{N+1} \frac{N(a_m)}{\bar{D}'(a_m)} e^{a_m \psi(u)} \quad \text{.. (4.48)}$$

Eqn. 4.48 above gives the functional form of $\psi(u)$ which when evaluated could be differentiated to get $\phi(\sigma)$. It can be seen that in a general case, $\psi(u)$ cannot be explicitly written as a function of u in which case a numerical approach is essential in proceeding further.

4.3 NUMERICAL TREATMENT IN MORE GENERAL CASES: SUGGESTIONS

In the example considered in Section 4.2 it was noticed that $\psi(u)$ that is to be evaluated (which when differentiated yields $\phi(u)$ or $\phi(\sigma)$, could not be obtained explicitly. In such cases, the relation

between $\psi(u)$ and u has to be numerically tabulated using Eqn. 4.48. The tabulated function when differentiated would yield $\phi(u)$ and further evaluation of $\Phi^*(\sigma)$ and $f(\sigma)$ according to Eqns. 3.9 and 3.10 of the preceding chapter is also to be carried out numerically. A resort to such numerical evaluation procedures should be commensurate with the accuracy warranted by the nature of the problem under consideration as compared to available metcods like that for example that of Weibull.

4.4 DISCUSSION

The present formulation and method of solution of the problem of size effects on strength is an attempt to pave the way to construct distribution functions suitable to a material behaviour as given by its size-mean strength relationship. The feasibility of such an approach is shown by getting the distribution function of Weibull starting from the corresponding size-mean strength relationship in a reparameterized form. However, construction of strength distribution functions suitable to more general forms of size-mean strength relations needs numerical approaches as was exemplified by the problem considered in Section 4.2. The scope of the present thesis is only to examine the feasibility of obtaining strength distribution function starting with a given form of

size-mean relationship with weakest link concept as basis, as this approach is more general in its application. Consequently no attempt is made to solve numerically the problem of size effects on strength for more general forms of size-mean strength relationships. Further investigations are necessary to obtain the strength distribution functions both in closed form as well as numerically, the size-mean strength relationships being obtained from tabulated data on strength of materials with specimen size as the variable.

CHAPTER FIVE

CHARACTERIZATION OF DIRECT TENSILE STRENGTH OF CONCRETE BY A DISTRIBUTION FUNCTION

5.1 INTRODUCTION

Concrete is a very important engineering material but an understanding of its properties and mechanical behaviour is still inadequate. Employing the classical concepts of continuum mechanics very little can be achieved in understanding the large scatter associated with concrete or the peculiarities in its mechanical behaviour. Considering the complex internal structure of concrete, it is easy to visualize the inherent difficulty in arriving at a suitable formal mathematical model for concrete. Inherently, concrete is a heterogeneous material consisting of mortar and aggregate phases mainly. The interface between the two phases being a zone of relatively weak bond, microcracks are present even at no load (89), due to shrinkage. The microcracks or interface bond failures act as stress raisers and cause progressive failure of the material under loading. While it is certain that concrete can hardly be idealized as brittle material because of the various forms (21), other than surface energy, in which the energy can be dissipated during crack

propagation, quantification of the various energy dissipation mechanisms appears to be hopelessly difficult at present.

In the present chapter an attempt is made to characterize the size dependent strength and scatter of concrete in direct tension, by a distribution function using the results of the previous chapter. Eventhough, concrete cannot be considered as a material that closely obeys weakest link concept because of the presence of various energy dissipating mechanisms, the decrease in scatter and mean strength with size prompts one to examine the applicability of the distribution function obtained based on weakest link concept.

In the following, a brief outline of the works of various investigations that brought to light the flaw sensitivity in concrete and the applicability of the concepts of fracture mechanics to concrete is given, so as to provide a basis for the subsequent development in the chapter wherein a study of the strength distribution of concrete is made.

5.2 BEHAVIOUR AND STRENGTH OF CONCRETE

The earliest proponent of the application of Griffith's theory to concrete is Kaplan (90), whose

work gave qualitative evidence of the applicability. Following the work of Kaplan, Glucklich (91), explained the difference in compressive and tensile strength of concrete, higher strength associated with smaller aggregate mixes and other related phenomena using the concepts of fracture mechanics. Glucklich explained the severity of the presence of microcracking around crack tips in concrete as well as the effect of aggregate phase in influencing the energy demand and dissipation curves during crack propagation in concrete. Romauldi and Batson (92), Sridhar Rao and Parimi (93) established the role of fibers in arresting crack propagation in fiber reinforced concrete. The works of the above investigators as well as those of Lott and Kesler (94), Welch and Haisman (95) have established that fracture mechanics concepts can be used to study the material properties of concrete. Further evidence regarding the effect of microcracking in concrete on the fatigue behaviour, cumulative damage and sustained load behaviour is due to Raju (96), Neal (97), Sushil Chandra (98).

Because the microcracks in concrete are random in their distribution in the material, scatter in test results is often present. However as has already been remarked, concrete cannot be rationally idealized as a

brittle material since catastrophic crack propagation and failure is not truely present as in materials like glass. Tension failures in concrete are relatively sudden and spontaneous as compared to compression tests. This is because the tensile stress field is more favourable for an arbitrarily oriented flaw to propagate as compared to compressive stress field, on the basis of which Glucklich (91), interpreted the ratio of the compressive to tensile strength of concrete. Further an examination of typical tests on concrete for tension (43) and compression (41) indicate that the mean strength and scatter decrease with increase in specimen size displaying that the features are akin to a system obeying weakest link concept. Applicability of Weibull's strength theory to concrete has been discussed by Popovics (99), who suggested that strength ratio of two specimens is related approximately to their volume ratio as given by

$$\frac{\bar{\sigma}_1}{\bar{\sigma}_2} = \left[\frac{V_2}{V_1}\right]^m \qquad \text{.. (5.1)}$$

where $\bar{\sigma}_1$ = mean strength of specimen of size V_1

$\bar{\sigma}_2$ = mean strength of specimen of size V_2

and m = a constant, the material parameter.

It can however be noticed that such a strength ratio as given by Eqn. 5.1 implies that the rate of decrease of strength with size is independent of specimen volume as well as an indefinite strength decrease with increase in size. This is a consequence of fact that the lowest mean strength of the material which is asymptotically approached as the specimen size increases has been assumed to be zero. Bolotin (72) discussed the applicability of the three parameter strength distribution function as given by Eqn. 2.16 discussed in Section 2.2.2 of the thesis, to the compressive and flexural strengths of concrete. Such a three parametered distribution function is essential to accomodate for the existence of non-zero lowest mean strength of a material. Consequently an attempt is made to characterize the scatter in direct tensile strength of concrete, by a distribution function using the results of the previous chapter.

5.3 EVALUATION OF MATERIAL PARAMETERS

In the following an attempt is made to find the material parameters σ_L, K and n as applicable to direct tension results of concrete, in the equation

$$\bar{\sigma} = \sigma_L \left\{ 1 + \frac{K}{V^n} \right\} \qquad .. (5.2)$$

where $\overline{\sigma}$ = the mean strength

V = specimen size

A comprehensive study of the effect of specimen size on the direct tensile strength of concrete is due to Kadleček and Špetla (43) and it is proposed to use the results of the authors in the present investigation. Kadleček and Špetla have eliminated the effect of slenderness ratio, viz the height to lateral dimension ratio and the results provide the volume effect on strength as observed in direct tension testing.

5.3.1 Details of Experimental Data

Kadleček and Špetla (43) have conducted tests in direct tension on cylinders and prisms; two series in each case with specimen volume as the variable. The details of the test specimens and the experimental results are given in Table 5.1 and Figs. 5.1 and 5.2. The last column of Table 5.1 contains the size-mean strength relations constructed in the present investigation and are explained in the following section. The Figs. 5.1 and 5.2 depicting the test data are reproduced from the paper of Kadleček and Špetla (43).

5.3.2 Material Parameters

Kadleček and Špetla suggested the size mean strength relations as given in Table 5.1 to be

TABLE 5.1: DETAILS OF EXPERIMENTAL DATA OF DIRECT TENSION TEST RESULTS ON CONCRETE

Test Series	Size-Mean Strength Relation: Originally given by Kadleček and Spetla (43)	Size-Mean Strength Relation: Constructed in the present Investigation
CYLINDERS[a]		
Series A[c]	$23.32\ V^{-0.041}$	$21\ \{1+ \frac{0.16}{V^{1.08}}\}$
Series B[d]	$24.78\ V^{-0.021}$	$23\ \{1+ \frac{0.085}{V^{0.5}}\}$
PRISMS[b]		
Series A[c]	$29.56\ V^{-0.03}$	$27\ \{1+ \frac{0.118}{V^{0.717}}\}$

[a] Cylinders of slenderness ratio 2

[b] Prisms of slenderness ratio 3

[c] Concrete with river pebbles of average compressive strength 300 kg/cm^2

[d] Concrete with granulated aggregates of average compressive strength 360 kg/cm^2

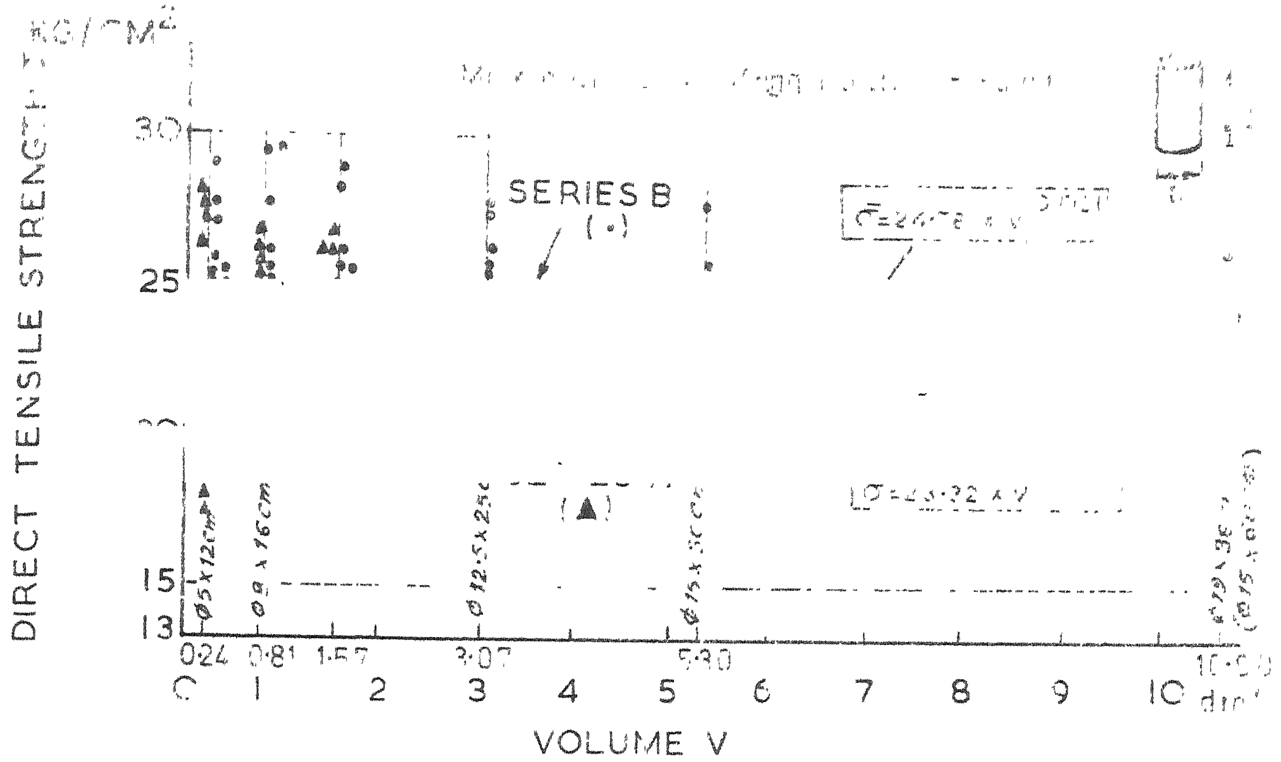


FIG.5·1 DIRECT TENSILE STRENGTH OF CONCRETE IN CYLINDERS (DIA D=5, 8, 10, 12·5 AND 15 CM OF CONSTANT SLENDERNESS H/D = 2) RELATIVE TO THEIR VOLUME, V (43)

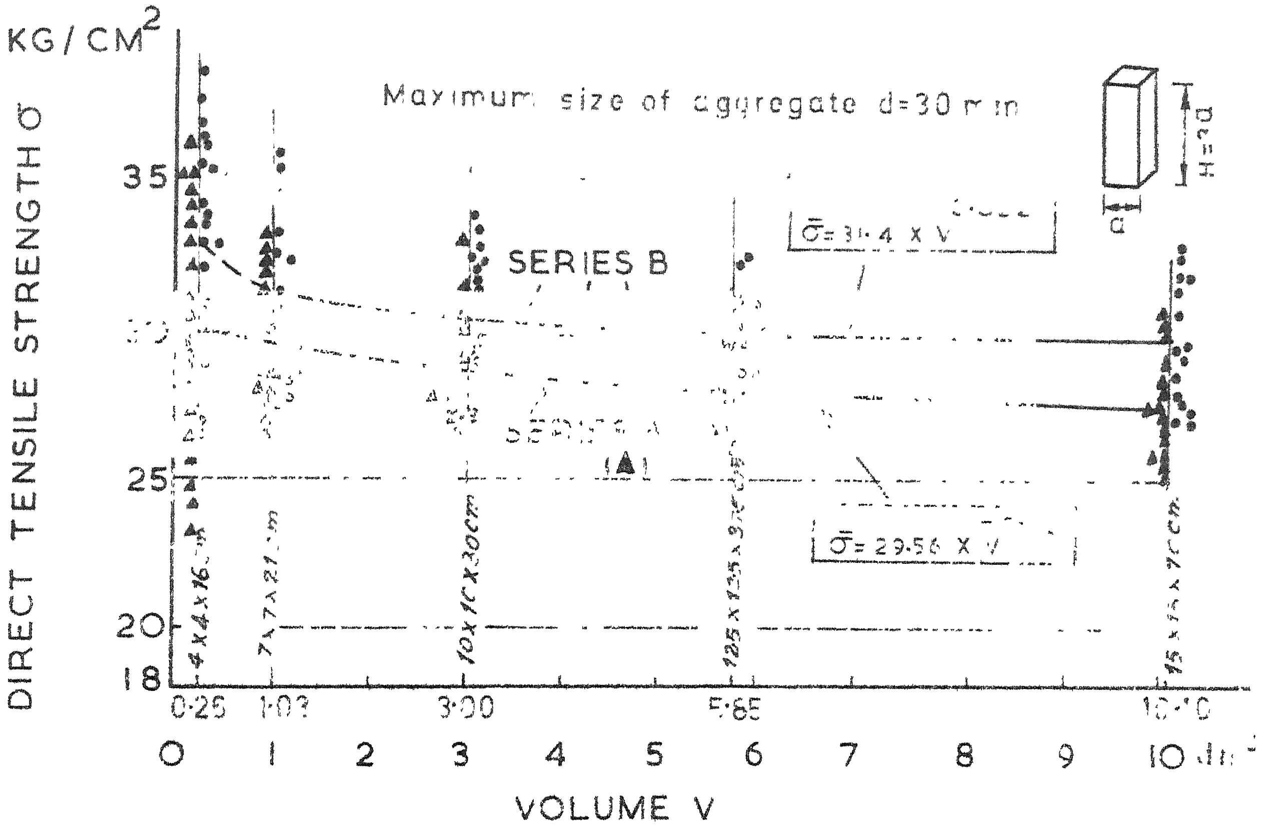


FIG.5·2 DIRECT TENSILE STRENGTH OF CONCRETE IN PRISMS OF (SIDE LENGTH=4, 7, 10, 12·5 AND 15 CM OF CONSTANT SLENDERNESS H/a = 3) RELATIVE TO THEIR VOLUME V (43)

representative of the test data. However the form in which they specified viz,

$$\bar{\sigma} = c_1 V^k \qquad .. (5.3)$$

where c_1 and k are constants

implies that as the specimen volume increases the tensile strength decreases indefinitely. This is contrary to the fact that the strength should approach a limiting value as the size increases as has been discussed in the earlier Section 5.2. Consequently an attempt is made to fit the data to an alternative form of size mean strength relation given by

$$\bar{\sigma} = \sigma_{L.} \{1 + \frac{K}{V^n}\}$$

which has been studied in detail in the earlier chapter for the associated strength distribution function. The material parameters are evaluated by trial and error as follows. The criterion for the choice of the set of values of σ_L, K and n is the closeness with which the theoretical strength distribution functions $\Phi^*(\sigma)$ and the one obtained from experimental data agree with each other at various volumes. However no quantitative measures of closeness have been attempted and the procedure is as follows.

From the Figs. 5.1 and 5.2 depicting the experimental data relating the specimen size and the failure strength, the experimental strength distribution function $\Phi^{*}(\sigma)$ corresponding to various values of σ is obtained from

$$\Phi^{*}(\sigma) = \frac{n_\sigma}{\bar{N}+1} \qquad .. (5.4)$$

where σ = is the stress level

n_σ = number of specimens failed upto a stress level σ

$\bar{N}$ = total number of test specimens.

The theoretical distribution function $\Phi^{*}(\sigma)$ is evaluated by using the result of chapter 4, as given by the Eqn. 4.22, viz,

$$\Phi^{*}(\sigma) = 1 - \exp\left[-V\left\{\frac{\Gamma(n+1)}{k\,\sigma_L}\right\}^{1/n} . (\sigma - \sigma_L)^{1/n}\right] \quad \text{for } \sigma > \sigma_L$$

$$= 0 \qquad \text{for } \sigma \leq \sigma_L$$

$$.. (5.5)$$

The quantities n, σ_L and K have been arrived at by trial and error so that the quantity $\Phi^{*}(\sigma)$ obtained from Eqns. 5.4 and 5.5 are in close agreement. The size-mean strength relation in terms of the material parameters σ_L K and n obtained by trial and error

as described above, in the case of cylinders (Series A and B), Prisms (Series A) are given in the last column of Table 5.1.

5.4 THEORETICAL AND OBSERVED DISTRIBUTION FUNCTIONS

Figs. 5.3 to 5.12 give the plots of $\Phi^*(\sigma)$ obtained experimentally by using Eqn. 5.4 and that obtained using Eqn. 5.5 with size-mean strength relation as given in the last column of Table 5.1. The Figs. 5.3 to 5.12 also contain the theoretically computed failure frequency $f(\sigma)$ given by Eqn. 4.23 viz,

$$f(\sigma) = \frac{V}{n} \left\{ \frac{\Gamma(n+1)}{K\,\sigma_L} \right\}^{1/n} (\sigma - \sigma_L)^{1/n - 1} \times$$

$$\exp \left[-V \left\{ \frac{\Gamma(n+1)}{K\,\sigma_L} \right\}^{1/n} (\sigma - \sigma_L)^{1/n} \right] \quad \text{for } \sigma > \sigma_L$$

$$= 0 \qquad \text{for } \sigma \leq \sigma_L \qquad .. (5.6)$$

Fig. 5.13 contains the failure frequency curves theoretically evaluated from Eqn. 5.6, corresponding to various volumes in the cases of cylinders (Series A and B). The plots of $\Phi^*(\sigma)$ and $f(\sigma)$ as given in Figs. 5.3 to 5.13 corresponding to various volumes of specimens are typical of the analytical study and as such only some of the results are presented. The plots indicate the increase in dispersion in strength with decrease in specimen volume.

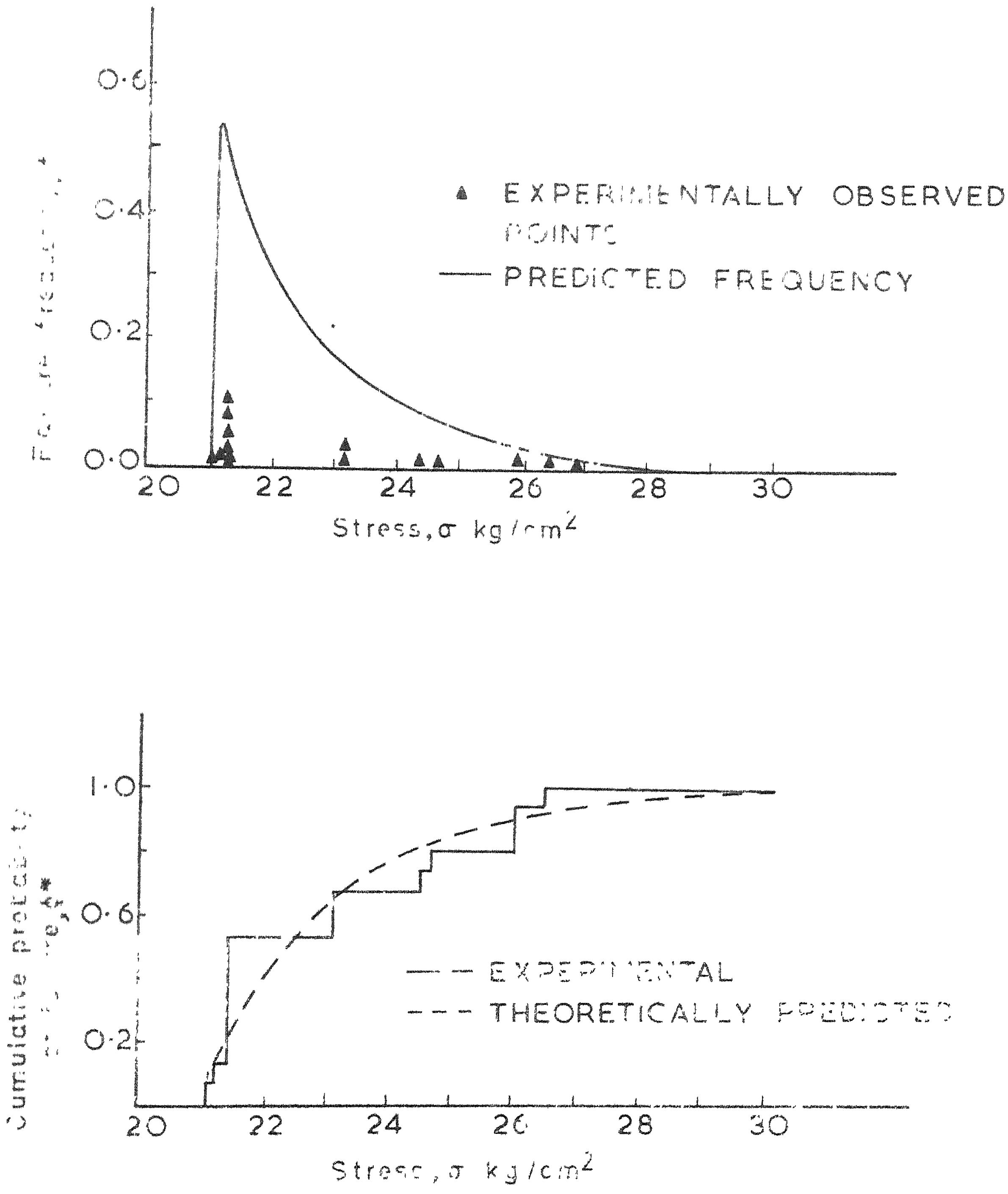


FIG. 5·3 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION CURVES (SERIES A. CYLINDERS OF VOLUME V= 1·57 dm^3)

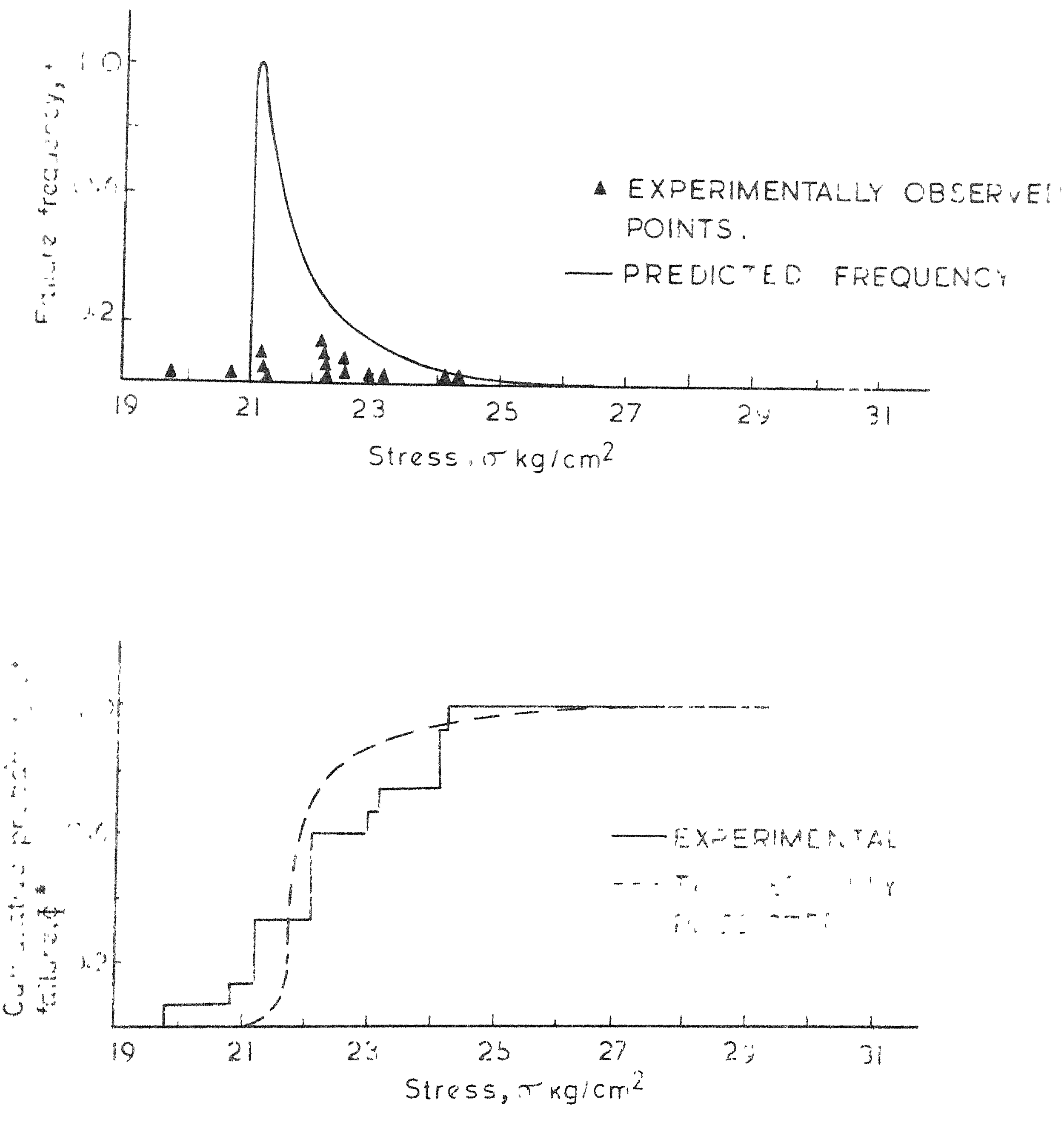


FIG. 5.4 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION CURVES (SERIES A, CYLINDERS OF VOLUME V=3...)

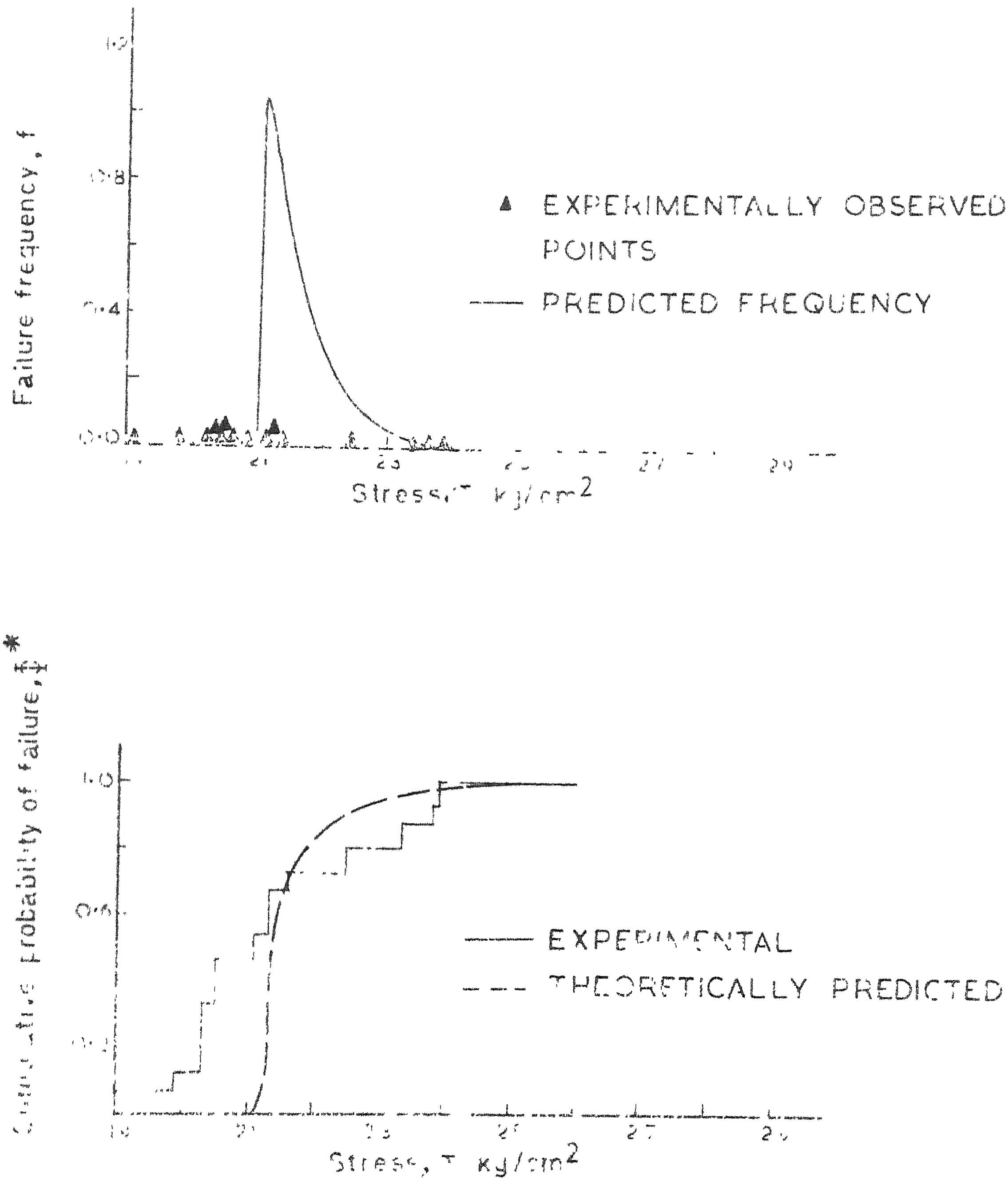
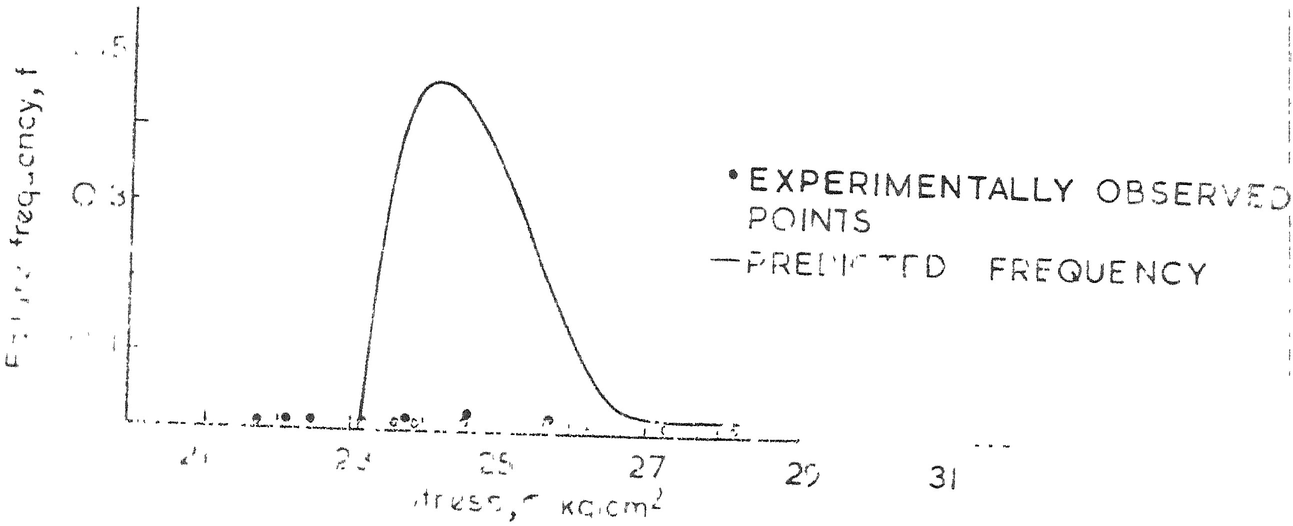


FIG. 5·5 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION CURVES (SERIES A, CYLINDERS [illegible]



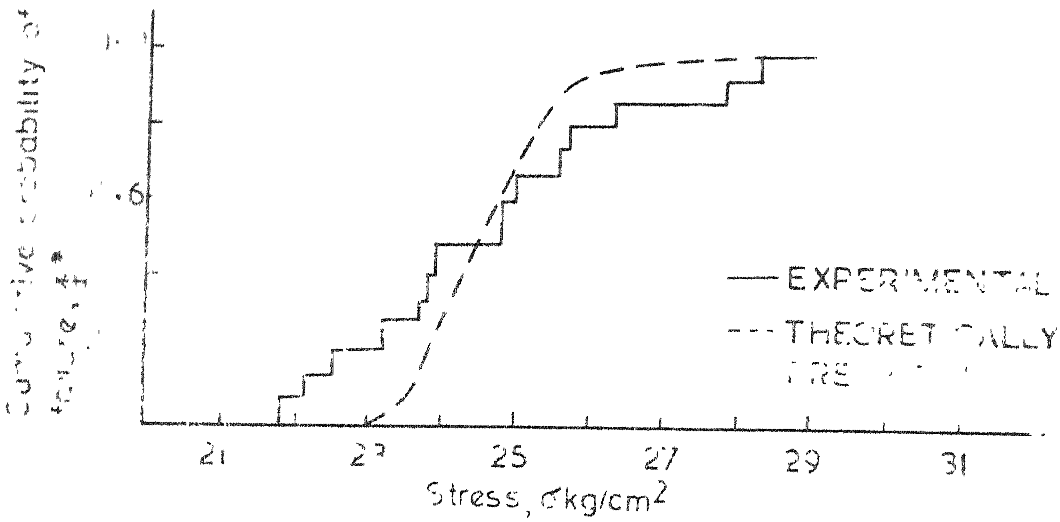
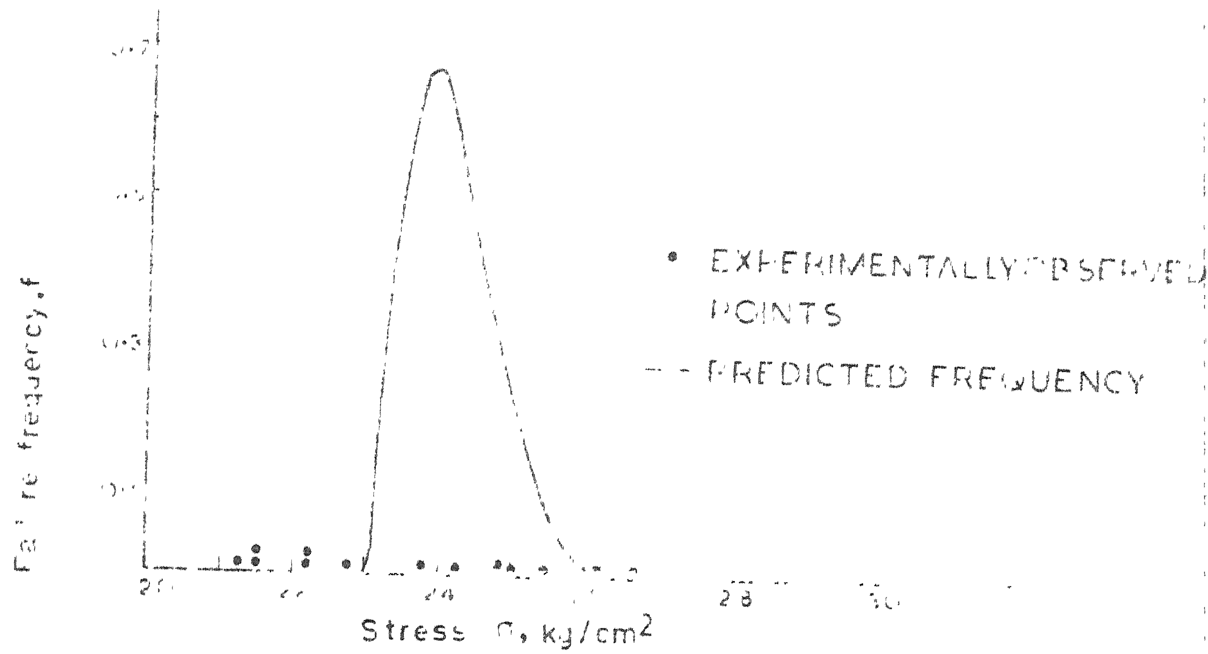


FIG. 5.6 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION CURVES (SERIES B, CYLINDERS OF VOLUME V=1.57dm³)



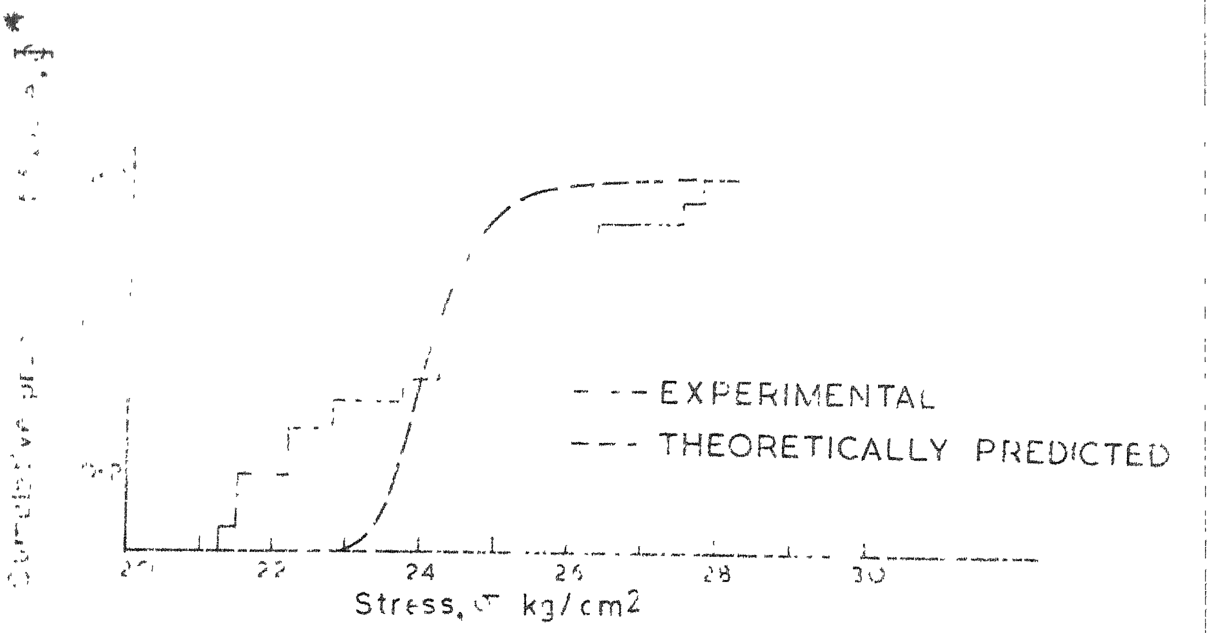


FIG. 5·7 FAILURE FREQUENCY PROBABILITY DISTRIBUTION CURVES (SERIES B, CYLINDERS OF VOLUME V=3…

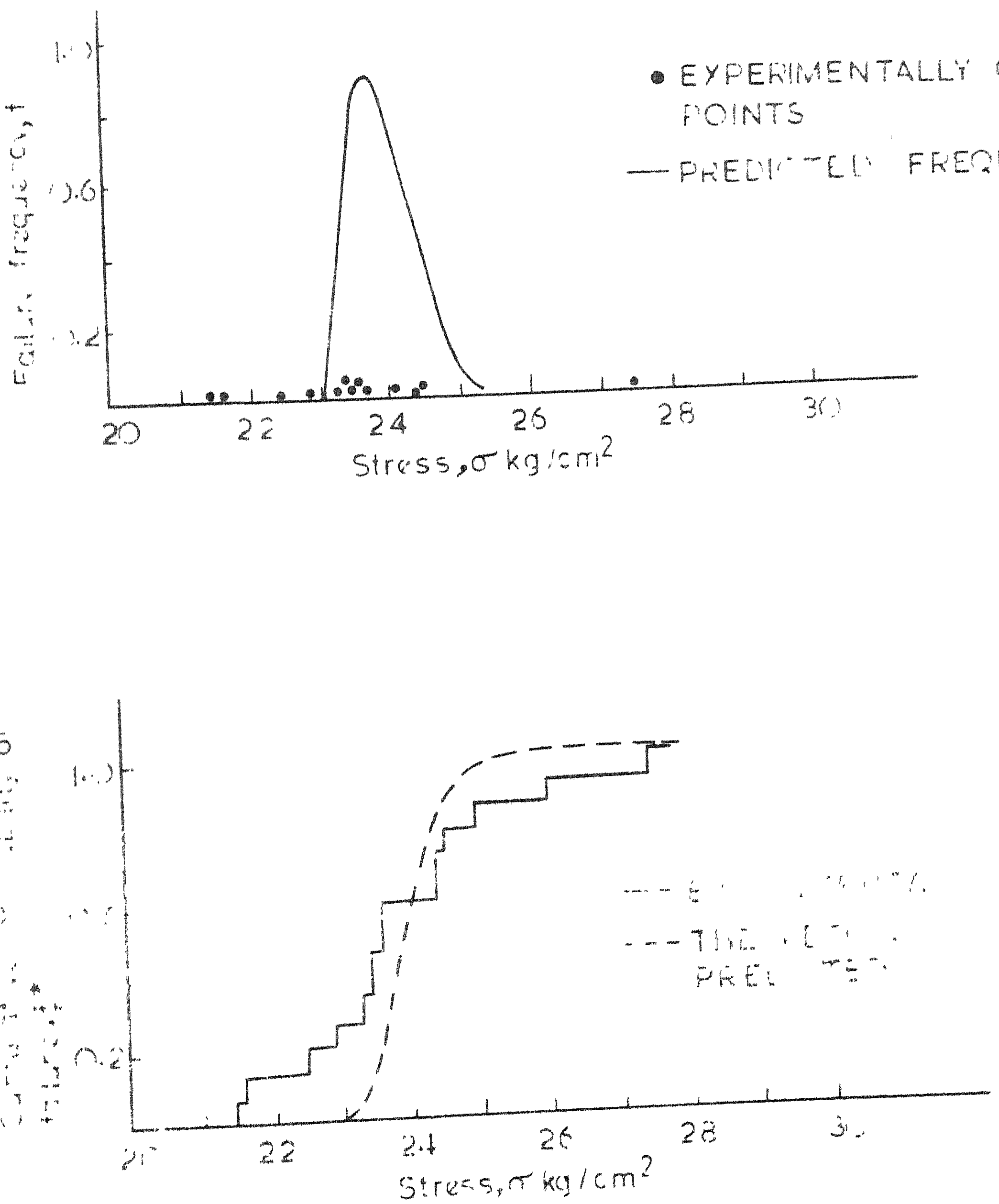
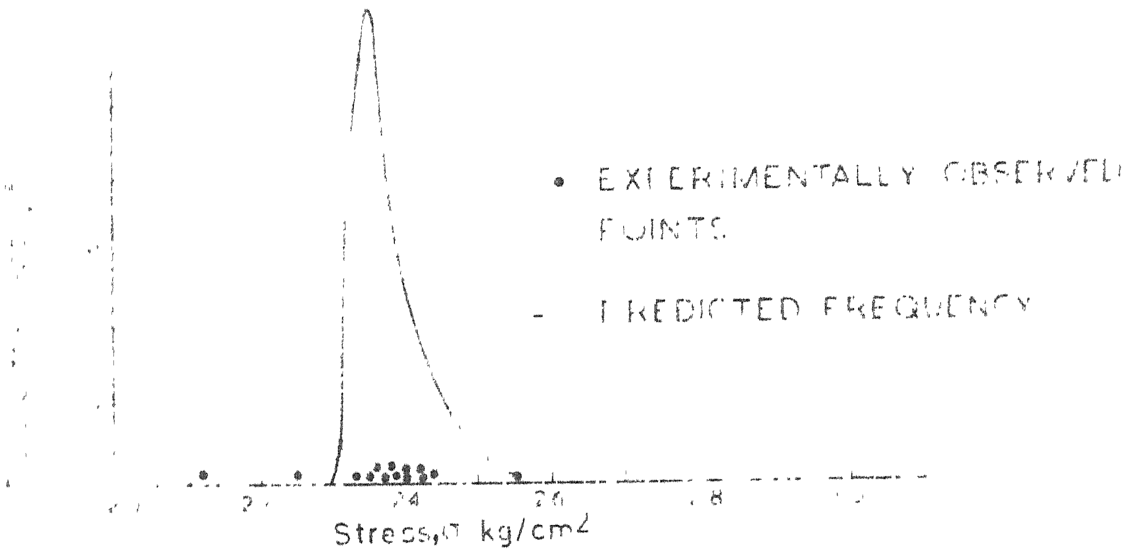


FIG 5.8 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION ...
SERIES B, CYLINDERS OF VOLUME V ...



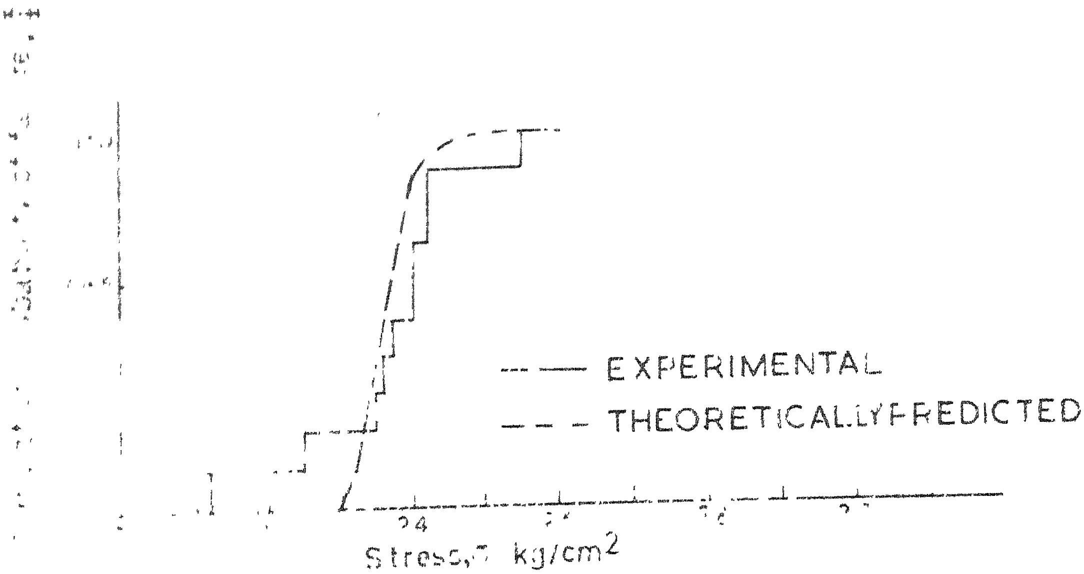


FIG 5.5 FAILURE FREQUENCY PROBABILITY DISTRIBUTION

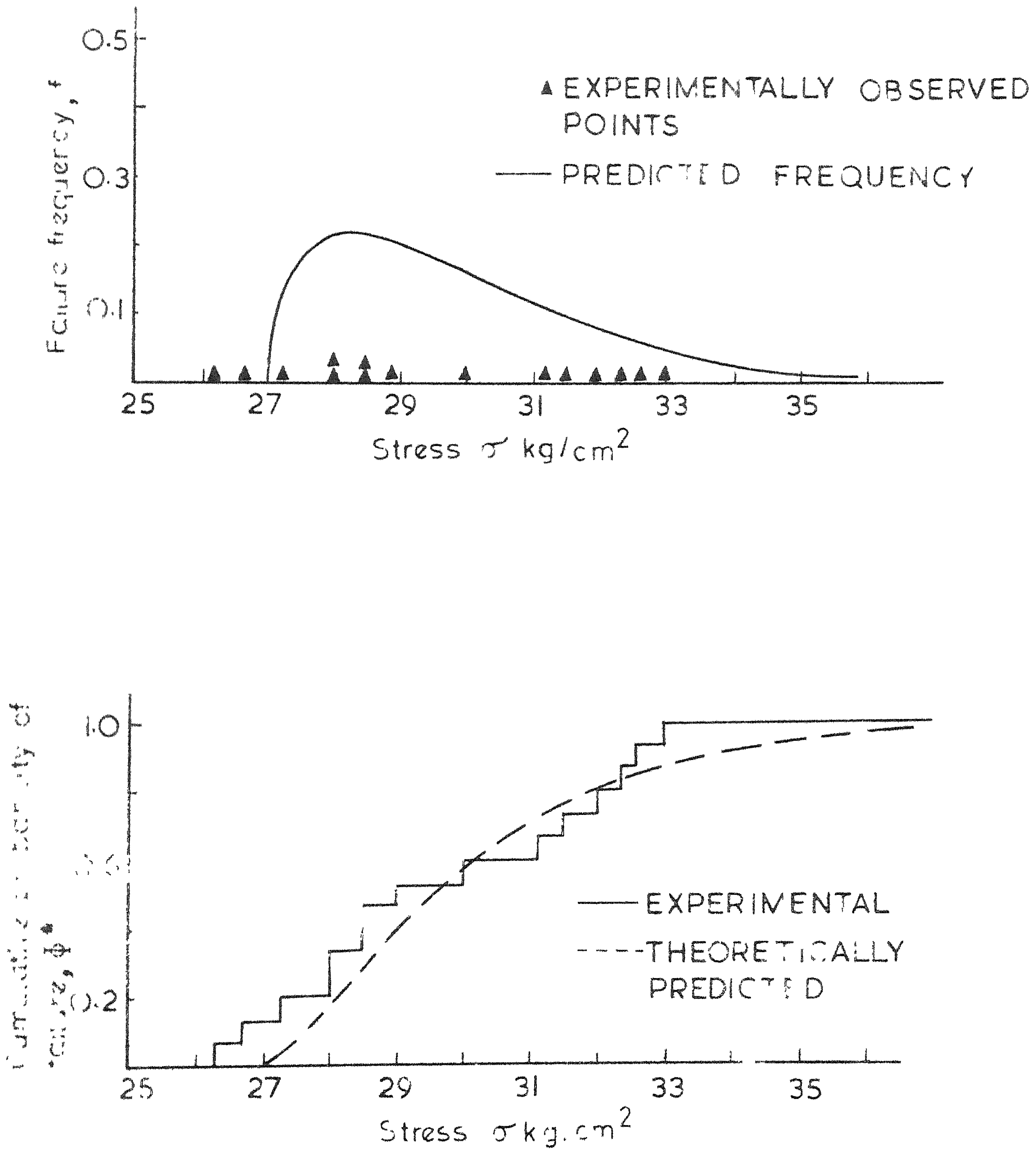
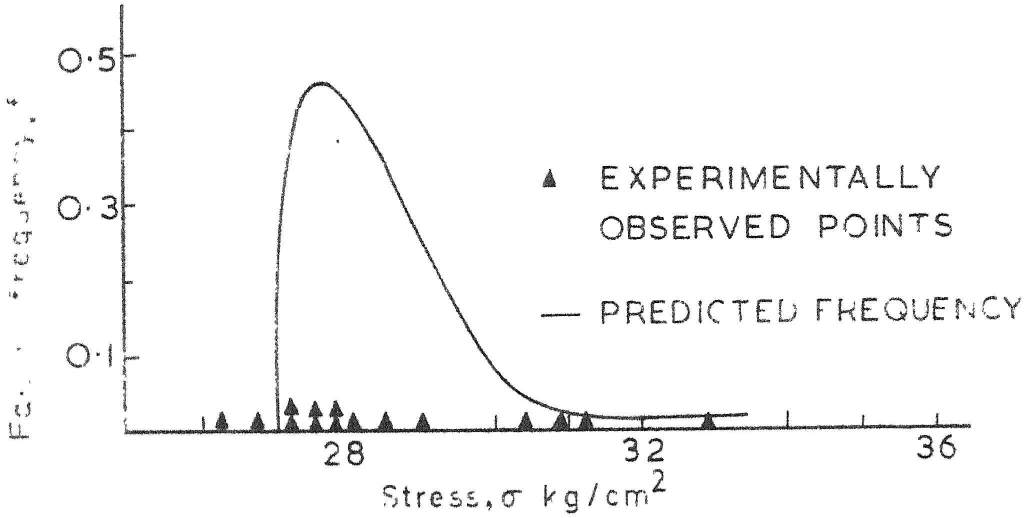


FIG 5.10 FAILURE FREQUENCY, PROBABILITY DISRIBUTION CURVES (SERIES A, PRISMS OF VOLUME V=.03 in³)



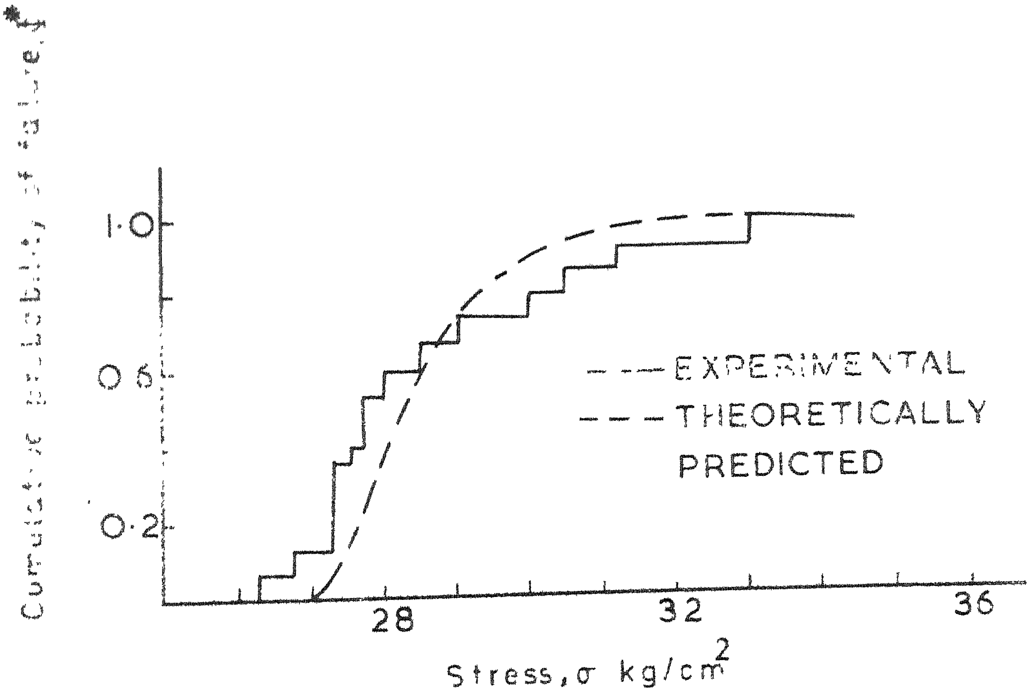
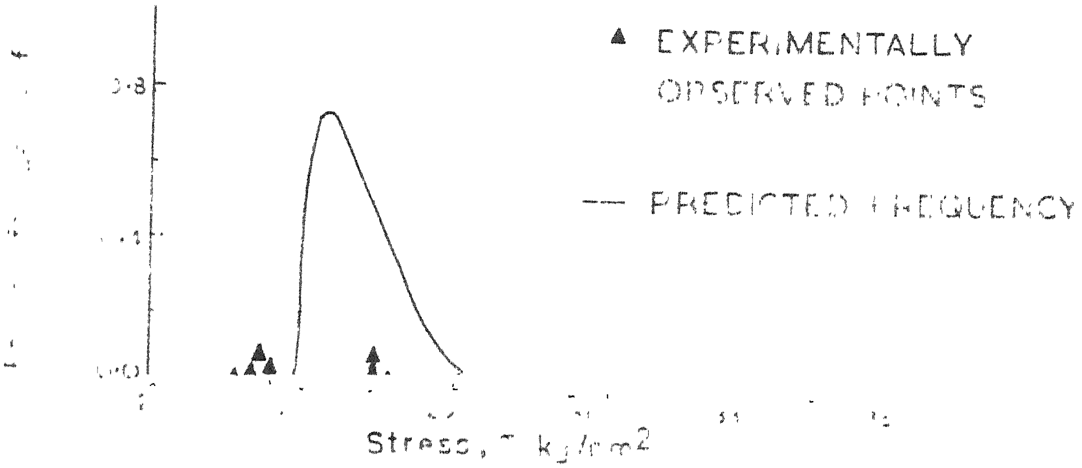


FIG. 5·11 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION CURVES (SERIES A, PRISMS OF VOLUME V=3·00 (



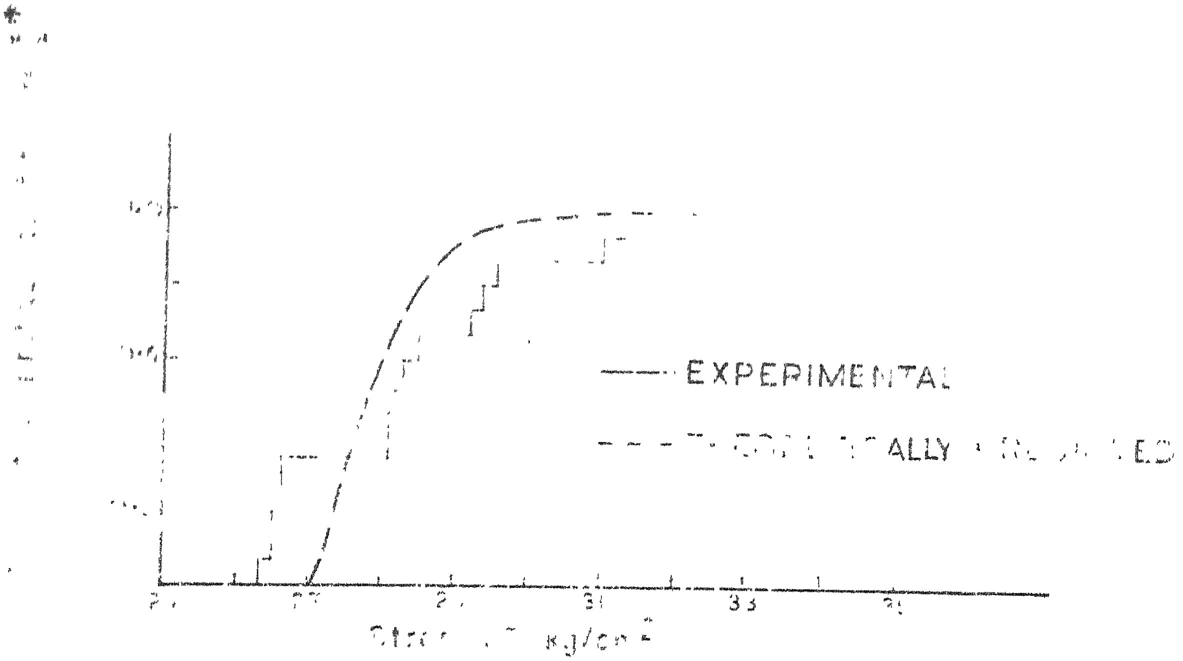


FIG. 5.12 FAILURE FREQUENCY, PROBABILITY DISTRIBUTION CURVES (SERIES A, PRISMS OF VOLUME V= 5.85 cm³)

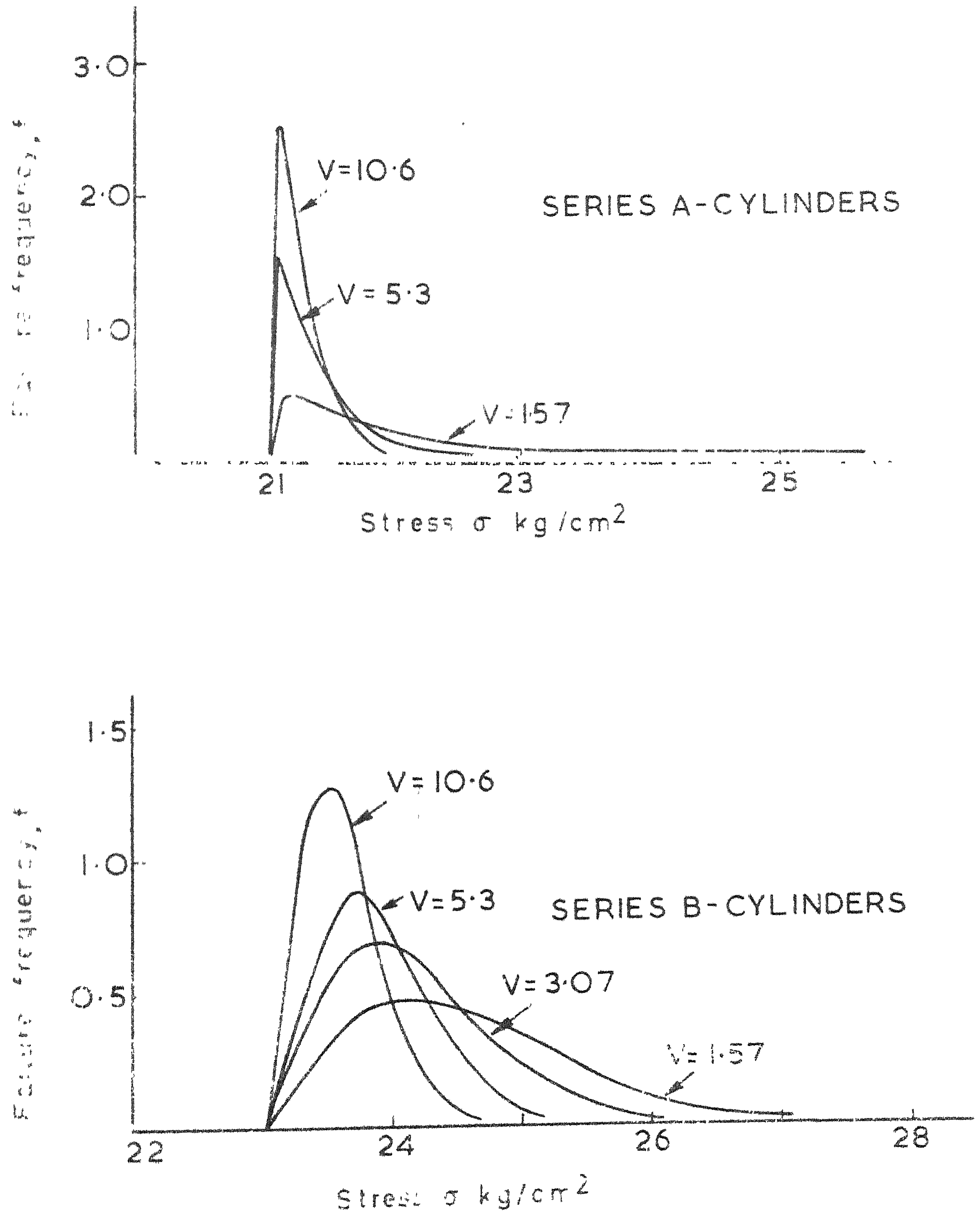


FIG. 5·13 FREQUENCY CURVES FOR FAILURE STRESS (THEORETICAL)

5.5 DISCUSSION

It can be seen from the plots of $\Phi^*(\sigma)$ (Figs. 5.3 to 5.12), the strength distribution functions obtained theoretically and experimentally are in agreement with varying degrees at various specimen volumes. Taking for example the results of Series A cylinders, Fig. 5.3, qualitatively the plots of $\Phi^*(\sigma)$ are in better agreement corresponding to cylinder volume $V = 1.57\ dm^3$ as compared to those in Fig. 5.4 and 5.5, indicating that the same values of the parameters σ_L, K and n at various specimen sizes are inadequate to characterize the scatter with same degree of accuracy.

It is interesting to compare the procedure adopted presently with that of Weibull as discussed in Chapter 2, Section 2.23. In the Weibull's method of plotting the curves $\Phi^*(\sigma)$, the lowest strength σ_u is arrived at by trial and error so that the logarithmic plot between $\log \frac{1}{1-\Phi^*(\sigma)}$ and $(\sigma - \sigma_u)$ is nearly a straight line. The values of the other parameters are fixed by the linear plot as discussed earlier in Section 2.2.3. Consequently in Weibull's plotting only one parameter σ_u is fixed apriori by trial and error and the others are fixed by Weibull's plot. In contrast, in the reparametrized solution as

used in the present chapter, all the three parameters are obtained by trial and error. Even though while using a computer the difference in computational effort between the trial and errorprocedure adopted presently and that in Weibull's graphical method may not be significant, Weibull's graphical method is systematic in its application. However in the present case an attempt is made to utilize the results of the theoretical results of Chapters 3 and 4 and as such the results are applied to the direct tensile strength of concrete. However, if a more general strength distribution function, other than that of Weibull is sought, the associated parameters are to be found only by trial and error, unless some systematic procedure is developed depending on the nature of distribution in any specific application.

CHAPTER SIX

SIZE EFFECT ON THE ERROR AND RELIABILITY IN PREDICTION OF STRENGTH IN MATERIALS TESTING

6.1 INTRODUCTION

The importance of materials testing in the development of physical concepts and in engineering design can hardly be overemphasized. A true understanding and interpretation of the results of tests on materials is essential in assessing their behaviour and in design applications. Usually testing is so planned and motivated, that the results are representative of the material characteristics in a more general situation, while the tests should be simple to perform. Consequently, material testing is basically predictive in nature. In majority of the cases, there are several factors that are to be accounted for before accurate inference could be drawn from test results. The factors arise because of nonfeasibility of easily simulating all conditions of the prototype structure in the tests such as, the load, environmental and geometrical conditions that are present in a prototype or the original structure in which the material behaviour is to be predicted. The various factors affecting strength

rupture tests or by splitting cubes or cylinders. Each of these show different strengths. The kinetic effects on material testing are manifested by the variation in strength with rate of loading. As has already been discussed in Section 1.3 of the thesis the size effects on strength are essentially due to occurance of flaws and imperfections in materials. The flaw occurance being random, the strength can be specified only by a distribution function. In such a case, it is of interest to assess the minimum number of test specimens that are to be tested so that the error in prediction of mean strength of the material does not exceed specified limits under different states of stress. For example flaws in concrete affect the tensile strength more than the compressive strength. Webster (120), studied the optimum sample size that corresponds to the minimum combination of testing as well as replacement cost, using information theory and Bayesian statistics.
In the study of Webster, the specimen size is not a parameter that influences the cost. Since the number of specimens to be tested for predicting the mean strength is dependent on the specimen size adopted, an assessment of the optimum specimen size that corresponds to minimum cost of testing programme is of interest. The

aforementioned problems are discussed in detail in the following sections.

6.2 SIZE AS A FACTOR IN SPECIFICATION OF MINIMUM NUMBER OF TEST SPECIMENS FOR GIVEN ERROR AND RELIABILITY IN PREDICTION OF STRENGTH

The size effect on strength of various material and the origins of the same are discussed in Section 1.3. The general feature of size effect is that the mean strength and associated scatter decrease with increase in specimen size. Consequently a problem of interest is to find the minimum number of test samples that are to be tested so that the population average strength can be predicted with a desired level of reliability and accuracy. Since the scatter or the coefficient of variation is affected by specimen size, the minimum number of test specimens to be tested also depends on specimen size. It is proposed to examine this problem making use of the test results of Kadleček and Špetla (43), on the tensile strength of concrete.

Since the basic problem of interest is to correlate the sample and population averages, Student's 't' distribution may be used as was discussed by Gordon (111), Benjamin and Cornell (112) and Hamilton (113) to correlate the error of the sample average, coefficient of variation and the number of tests as given by the

equation

$$N = \left(\frac{t\ C_v}{E}\right)^2 \qquad .. (6.1)$$

where N = number of tests

t = value of Students 't' for N-1 degrees of freedom at a specified level of probability

C_v = coefficient of variation

and E = maximum percentage error of the sample average

The use of Student's t distribution as given by Eqn.6.1 implies that the distribution of strength of the material is Gaussian with the coefficient of variation not known. Also it is assumed that the best estimate of the coefficient of variation is the same as that given by the coefficient of variation of the sample. Results of Kadleček and Špetla (43) from direct tension tests on concrete cylinders and prisms are shown in Table 6.1. The table gives the details of specimen sizes, average strength, coefficient of variation and the number N of test specimens calculated according to Eqn. 6.1 corresponding to an error E of 5% with 90% probability. While calculating by Eqn. 6.1 for N, the value of 't' is taken from the standard table of Student's distribution (111,113), so that the value of N and t are consistant; i.e. the number of degrees of freedom N-1,

TABLE 6.1: NUMBER OF TEST SPECIMENS (N) REQUIRED TO BE TESTED TO ENSURE AN ERROR OF 5% WITH 90% PROBABILITY IN ESTIMATING THE MEAN STRENGTH, AS AFFECTED BY SIZE OF THE TEST SPECIMENS (DIRECT TENSION TESTS)

Volume dm^3	Size* cm	Series A Average strength Kg/cm^2	Series A Coefficient of variation %	Series A N	Series B Average strength Kg/cm^2	Series B Coefficient of variation %	Series B N
CYLINDERS							
0.24	5 x 12**	22.8	15.2	26	23.5	12.6	21
0.81	8 x 16	23.4	9.1	11	24.8	10.6	14
1.57	10 x 20	23.2	8.2	9	24.7	7.8	9
3.07	12.5x 25	22.2	6.0	6	24.2	8.1	9
5.30	15 x 30	21.5	7.2	7	23.9	6.7	7
10.60	19 x 38	21.3	7.0	7	23.6	5.5	5
PRISMS							
0.26	4x4x16**	30.0	13.2	21	33.0	10.4	14
1.03	7x7x21	29.6	7.5	8	31.0	9.1	11
3.00	10x10x30	28.4	6.4	7	30.7	6.7	7
5.85	12.5x12.5 x37.5	28.1	5.7	5	28.9	4.4	4
10.10	15x15x45	27.5	6.1	6	29.6	6.8	7

* diameter x length in the case of cylinders and side x side x length in the case of prisms

** aspect ratio is different from the rest in the set

Series A: Average compressive strength 300 Kg/cm^2

Series B: Average compressive strength 360 $Kgcm^2$

is so chosen that the value of N and the value of 't' from the Student's distribution table give a value of the coefficient of variation C_V from Eqn. 6.1, very close to that given in Table 6.1 for various cases.

It can be noted from the values of N calculated and given in Table 6.1, that the number of specimens to be tested for a given accuracy in prediction of strength varies with size. A smaller number of tests are necessary if larger specimens are used, than those required for smaller specimen sizes. Also the number varied with specimen shape and aggregate type, as a consequence of the change in the coefficient of variation. It is remarked that for a given specimen size, different number of sample tests are required in tension and compression for the same error and reliability in prediction of mean strength for concrete because the flaw sensitivity of the material and consequently the coefficient of variation varies with state of stress.

For an illustration of the above remark the minimum number of specimens N, required to be tested to predict the mean compressive strength of concrete, calculated using the results of Rajendran (41), may be used. Table 6.2 gives the details of test data of Rajendran and the values of N obtained by the application

of Student's t distribution through Eqn. 6.1, as before.

TABLE 6.2: NUMBER OF TEST SPECIMENS (N) REQUIRED TO BE TESTED TO ENSURE AN ERROR OF 5% WITH 90% PROBABILITY IN ESTIMATING THE MEAN STRENGTH AS AFFECTED BY SIZE OF THE TEST SPECIMENS (DIRECT COMPRESSION TESTS)

Volume Cu"	Size in.	Average strength lb/sq"	Coefficient of variation %	N
64	4 x 4 x 4	4654	2.936	3
216	6 x 6 x 6	4450	2.813	3
512	8 x 8 x 8	4288	2.799	3
1000	10 x10 x 10	4088	2.714	3

It can be seen from Table 6.2 that the minimum number of test specimens did not differ corresponding to various specimen volumes as the decrease in coefficient of variation with increase in specimen size is very insignificant. Also comparing the values of 'N' from Table 6.1 and 6.2, it may be seen that more specimens are required to be tested in tension than in compression for the same error and reliability in predicting the corresponding mean strength. It should be noted, that this observation is by no means perfectly

conclusive since the material as well as the specimen volumes considered in Tables 6.1 and 6.2 corresponding to those of Kadleček and Špetla (43) and Rajendran (41), are entirely different.

Studies of Johnson (40) indicate that for a given mix of concrete, and for a given specimen size, the coefficient of variation is more in indirect tension tests than in compression tests. Consequently it may be inferred that more specimens are required to be tested in tension than in compression, to specify mean strength with a given reliability and error in prediction.

However, since the state of stress in indirect tests is not purely tensile, the equality of specimen volumes in compression and in indirect tests does not ensure perfect identity of test conditions. Since the material parameters in the strength distribution of a material are characteristic of the material, the coefficient of variation under different stress states should be expressible in terms of the parameters.

6.3 OPTIMUM SPECIMEN SIZE AND SAMPLE NUMBER IN MATERIALS TESTING

Since the mean strength and the scatter or coefficient of variation of strength generally decrease with increase in specimen size, the number of test specimens N, to be tested to predict the mean strength of the material with a given error and reliability level varies

with the size of test specimens. As discussed in the preceding sections of the chapter, as the coefficient of variation decreases, the number of specimens to be tested decreases; qualitatively, smaller the specimen size more the number to be tested. Apparently, in any programme of testing, one is confronted with a decision on the specimen size and the number of specimens so that the decision is optimal. The criterion of optimality could be with respect to the total cost of testing which is the sum of the material cost and cost incurred in testing the specimens. Since, by adopting larger specimens (i.e. more material and more material cost), smaller number is required (i.e. less number of specimens and less testing cost), it is of interest to find the optimum sample number and specimen size so that the total cost of testing is minimized. This problem is examined in this section.

The relation between minimum number of test specimens N and the coefficient of variation C_V is given by Eqn. 6.1 (as discussed in Section 6.2). The coefficient of variation, in general decreases with specimen volume V. The total cost of testing may be written as

$$C_T = (K_1^* V + K_2^*) \, N \qquad .. (6.2)$$

where C_T = total cost of testing in Rs.

K_1^* = cost of unit volume of the material in Rs.

V = specimen volume

N = number of specimens

and K_2^* = cost of testing a single specimen.

The quantity N in Eqn. 6.2 is dependent of specimen volume V and the coefficient of variation in strength C_v, through Eqn. 6.1. The possibility of an optimum specimen volume V existing is apparent, since by adopting specimens of large volume, the material cost increases, while the testing cost decreases; by adopting smaller specimens material cost decreases while the testing cost increases. The exact variation of total cost C_T with specimen size V depends, however, on the values of K_1^* and K_2^* in Eqn. 6.2.

Considering for example the direct tension testing of concrete, the number of test specimens N, required to be tested to ensure an error of 5% , with a probability of 0.90 in estimating the mean strength is given in Table 6.1, using the results of Kadleček and Špetla (43). To examine the total cost of testing as dependent on specimen volume adopted, the values of K_1^* and K_2^* in Eqn. 6.2 are assumed and are as follows (for purposes of illustration).

$$K_1^* = \text{Rs. } 0.5 \text{ per dm}^3 \text{ of concrete} \quad .. (6.3)$$
$$K_2^* = \text{Rs. } 5 \text{ per test specimen} \quad .. (6.4)$$

so that

$$C_T = (0.5\ V + 5) \quad N \quad .. (6.5)$$

In specifying the cost of material through K_1^*, approximate cost of 'glue' necessary to fix the specimens to the end plates, so that the specimen could be tested in direct tension, is also included. Since the values of 'N' for various specimen volumes 'V' are already calculated according to Eqn. 6.1, and are given in Table 6.1, the total cost C_T may be evaluated using the values of K_1^* and K_2^* as given in Eqns. 6.3 and 6.4. The total cost C_T according to Eqn. 6.5 is evaluated for the data in Table 6.1 (corresponding to series A and B cylinders and series A and B prisms) and the results are plotted in Figs. 6.1 to 6.4. The plots show the variation of total cost C_T in Rs. with specimen volume V(dm^3). In plotting the Figs. 6.1 and 6.2 the cost corresponding to cylinder volume 0.24 dm^3 is not considered since the cylinders have a different slenderness ratio from the rest of the specimens as is indicated by a double asterisk in Table 6.1. Similarly, the cost of testing prisms of volume 0.26 dm^3 is not considered in plotting Figs. 6.3 and 6.4. The points

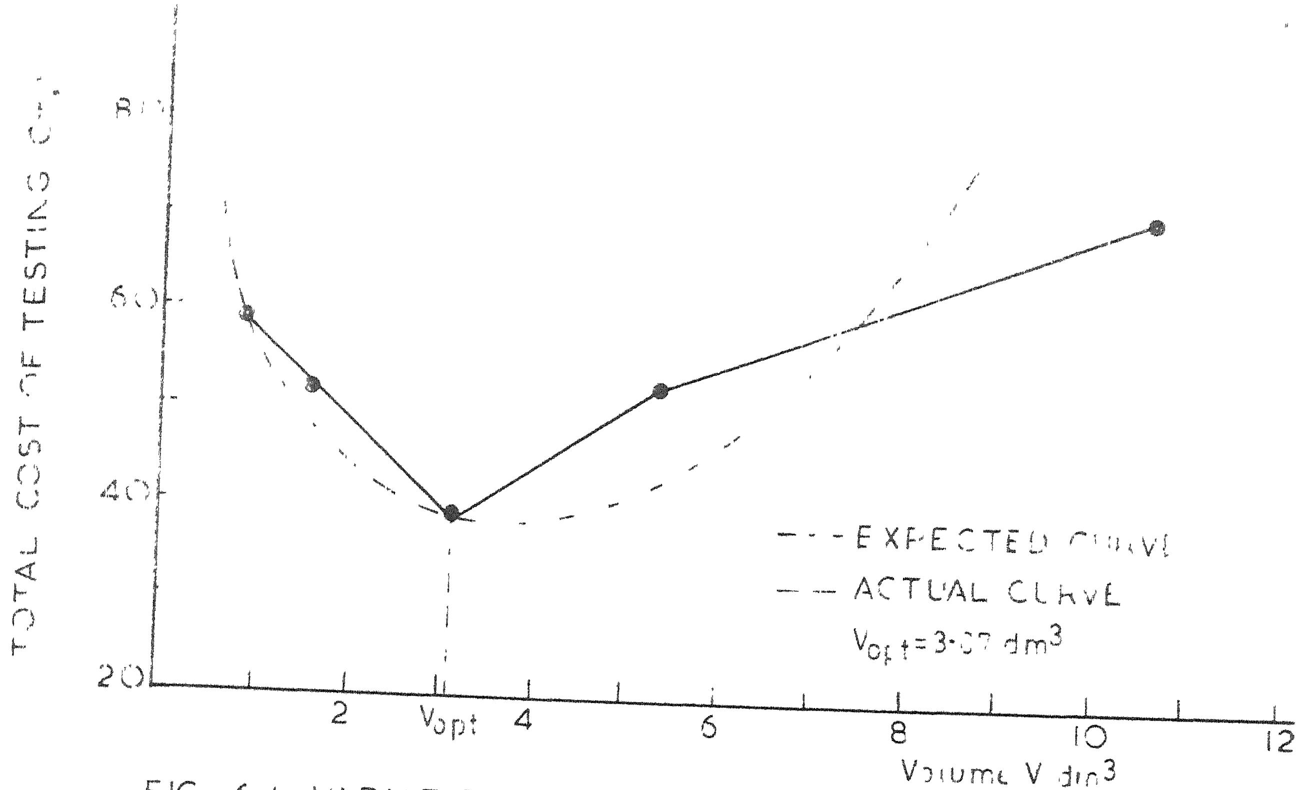


FIG. 6·1 VARIATION OF TOTAL COST OF TESTING WITH SPECIMEN VOLUME (SERIES A, CYLINDERS)

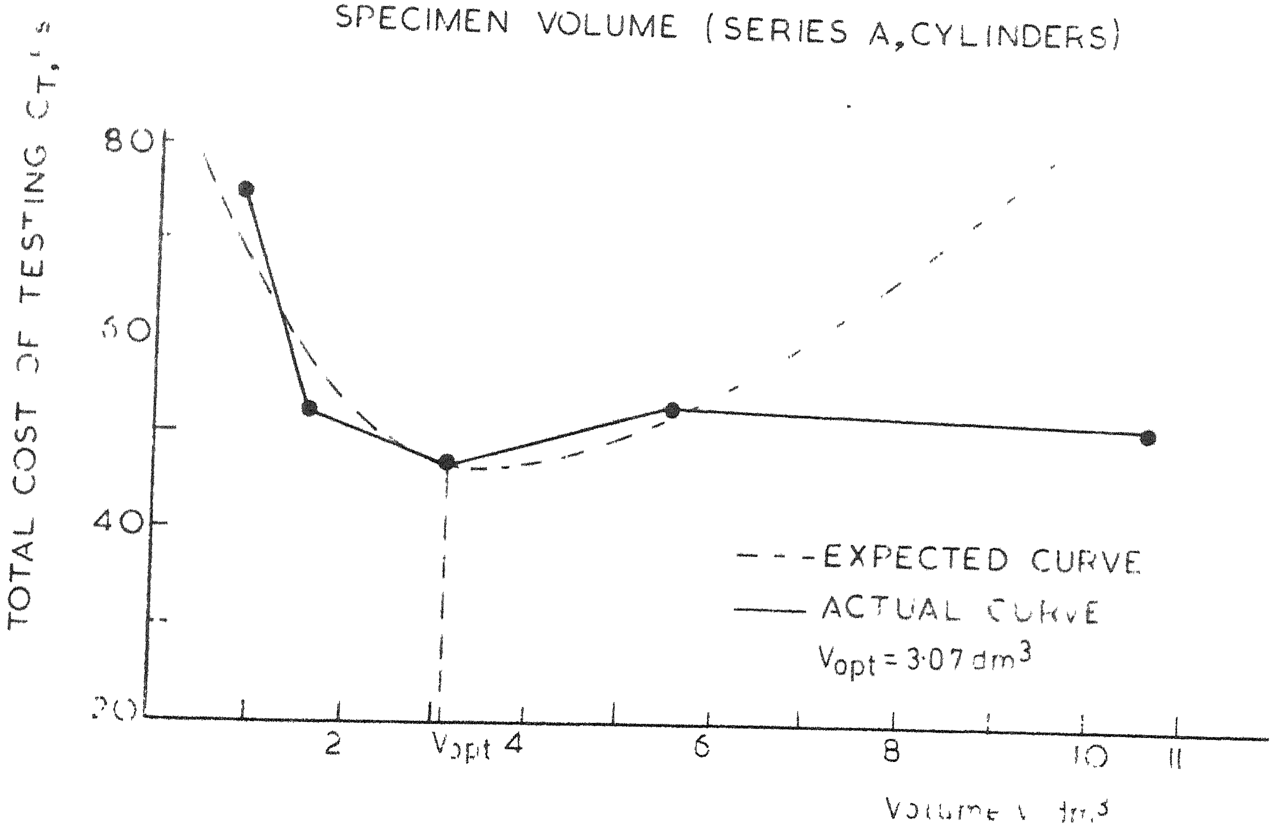


FIG. 6·2 VARIATION OF TOTAL COST OF TESTING WITH SPECIMEN VOLUME (SERIES B, CYLINDERS)

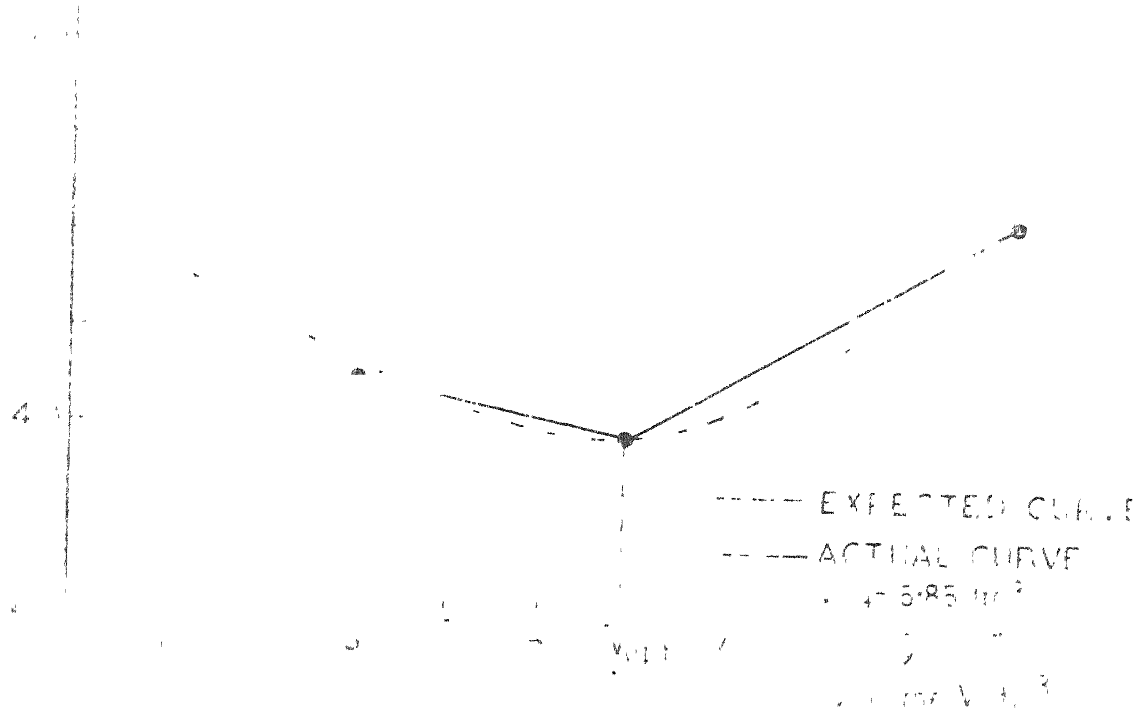


FIG. 6.3 VARIATION OF TOTAL COST OF TESTING WITH SPECIMEN VOLUME (SERIES A, PRISMS)

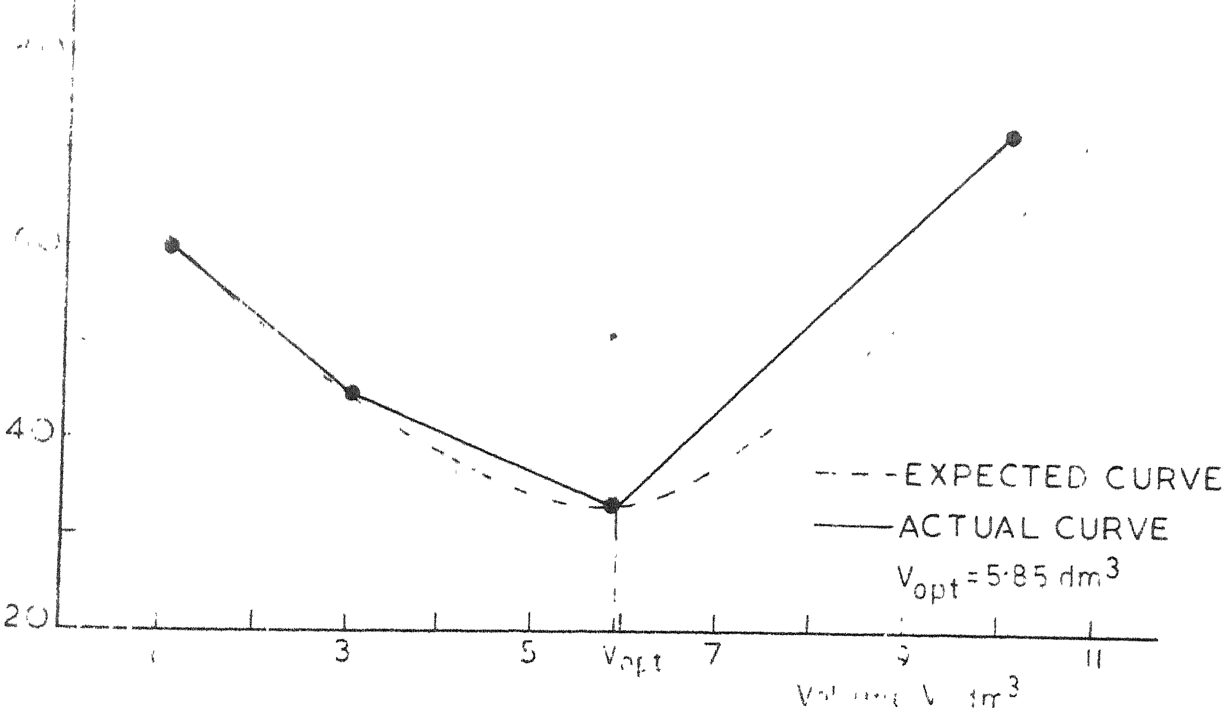


FIG. 6.4 VARIATION OF TOTAL COST OF TESTING WITH SPECIMEN VOLUME (SERIES B, PRISMS)

corresponding to actual calculated values of C_T for various volumes in the figures are joined by straight lines eventhough they could possibly lie on convex curves, as shown in dotted(profile only) if a consistant decrease in the coefficient of variation of strength is present with increase in specimen volume (as is theoretically expected). The presence of local maxima in the plots of the variation of C_T with V in Figs. 6.2 and 6.3 is a consequence of the higher value of the coefficient of variation of strength corresponding to that volume in the experimental results (possibly the limitations of accuracy of the experimental results). However, in case a consistant decrease in coefficient of variation of strength with increase in specimen volume is present (as is theoretically expected) such local maxima will not be present. From Figs.6.1 and 6.2 it can be seen that the optimum volume of cylinders is 3.07 dm^3 and the optimum prism volume is 5.85 dm^3 for the problem considered using the results of Kadleček and Šeptla. The corresponding sample sizes 'N' are as given in Table 6.1 corresponding to the optimum specimen volumes.

6.4 DISCUSSION

In this chapter some possible practical applications of brittle fracture statistics are discussed. While correlating the strength from tests to that in actual

structures, it is to be noted that, especially in materials like concrete, quality control exercised in manufacture and placing can significantly affect strength as well as coefficient of variation (114 to 119). For example, the test results of Kadleček and Špetla (43) and Rajendran (41) considered in the present study are for concrete manufactured and placed under laboratory conditions and consequently the quality control exercised can be expected to be good. However, the same concrete under field conditions may not reproduce the same scatter characteristics. Nevertheless, the size effect on strength is prevalent in any case and the study carried out, to find an optimum specimen size for minimum cost of testing will be useful in planning a test programme economically. The basic principles underlying the example considered in the present study can be applied to other materials as well in standardizing the test programmes. Even though the assumption of normal distribution of strength in assessing the number of specimens applying Student's distribution is approximate (especially as applied to materials that exibit a significant skew in their strength distribution), it provides a simple means of fixing the number of test specimens economically in planning a test programme.

Through the study of Johnson (40) indicating the existance of larger coefficient of variation in indirect tension testing of concrete than in compression (for same specimen size and mix quality) and from the present study as is discussed in this chapter based on the calculations of Tables 6.1 and 6.2, more specimens are required to be tested in tension than in compression to specify the mean strength with a given error and reliability in prediction. However, as already remarked, due to the presence of secondary stresses in indirect tension, the equality of specimen volumes does not ensure identity of test conditions in compression and indirect tension. Further studies towards analytically obtaining the coefficients of variation under different stress states (like splitting of cylinders etc.) in terms of the basic material parameters, are necessary. Such studies in conjunction with those on the actual mechanisms of fracture involved under different states of stress in materials would give further insight in the statistical aspects of strength of materials as well as in consequent practical applications.

CHAPTER SEVEN

A PHENOMENOLOGICAL THEORY OF FRACTURE BEHAVIOUR OF CONCRETE - LIKE MATERIALS

7.1 INTRODUCTION

In the preceding chapters of the thesis, the phenomenon of size effect on strength and associated scatter, is studied in some detail. Of related interest is the phenomenon of size effect on mechanical behaviour in materials. While size effects on fatigue strength and ductile-brittle transition in materials are known to be prevalent (58,15) as discussed in Section 1.4 of the thesis, systematic enquiries into the various aspects of size effects on mechanical behaviour of materials are scanty. This observation is somewhat surprising in the wake of the fact that size effect on strength has been so extensively studied and one could always conjecture logically that the same features of material structure like flaws, which affect strength could contribute in some measure to affect the other aspects of mechanical behaviour of materials as well. Like size effect on strength, size effect on ductile-brittle transition, size effect on stiffness etc., are not encompassed by

classical theories of mechanical behaviour of materials. These phenomena cannot be rationally interpreted without resort to the study of inherent imperfections in materials. An insight into various aspects of size effects on stiffness, nonlinearity in stress-strain behaviour, ductile-brittle transition etc., is essential in correlating the model and prototype tests, in addition to being of fundamental value in understanding the behaviour of materials. Consequently an attempt is made in this thesis to study some aspects of this problem.

In the following an attempt is made to propose a unified theory that correlates the phenomena of size effects on strength and size effects on stiffness and ductility in materials. The theoretical development is applicable to materials in which a progressive breakdown in the internal structure of the material due to growth of microcracks is present. Also in the theoretical development, time, or temperature effects and multiaxial stress conditions are not considered. In the formulation of the theory, some results of the theory of composite materials are invoked and consequently in the following sections a brief review of the background and implications of the results invoked is carried out for an appraisal

of the present contribution.

7.2 THEORY OF COMPOSITE MATERIALS

Though composite material systems like mortars and concrete are known to be in wide use from very early stages of technical activity, the increased use and innovation in several new composite materials like alloys, fibre reinforced plastics etc. of late has triggered active research into this area (101). While the experimental and theoretical studies on the mechanical behaviour of composite materials are of interest in assessing their properties for engineering application, the subject is also of fundamental value since the related concepts provide a means of assessing the overall mechanical properties of a material system in terms of its microstructure. This aspect can be appreciated by observing the gross behaviour of a polycrystalline aggregate. A polycrystalline material may be visualized as a complex material consisting of several randomly oriented grains, and knowing the properties of individual grains the behaviour of the aggregate may be assessed (100). The basic problem of interest in the theory of composite materials is to arrive at an estimate of the properties of the material in terms of those of the constituent materials and

the details of the geometry of the constituent phases. The following section briefly outlines methods of estimating the elastic moduli of a composite material.

7.2.1 Methods of Evaluating the Elastic Moduli

To estimate the elastic moduli of a composite material, the straight forward approach is to measure geometrically the relative locations and extents of the various constituent phases with known material properties and solve the field equations of elasticity in each phase and match solutions on the phase boundaries. While such an approach will be useful in studying materials where there is a definite pattern in phase disposition like in layered media, the same is practically impossible in the case of complex materials like concrete. In addition, in such materials the phase disposition is essentially random and for a formal approach to the study of such media, an appeal must be made to statistical continuum theories (102). In view of the fact that study of composite materials through a statistical theory is formidable in many cases, one is forced to go in for what are known as effective moduli of the composite material which are representative of the average behaviour of the composite material. Formally (100) the concept

of effective moduli can be given by

$$\bar{\sigma}_{ij} = \lambda^{*} \bar{\Delta} + 2 \mu^{*} \bar{\epsilon}_{ij} \qquad \ldots (7.1)$$

where $\bar{\sigma}_{ij}$ = space average stress

$\bar{\epsilon}_{ii}$ = space average strain

λ^* and μ^* = effective Lame's parameters

and $\bar{\Delta}$ = space average dilatation

Paul (103) and Hashin (104) derived bounds on the effective elastic moduli E^* and G^* using energy principles of elasticity theory. While fixing the bounds on elastic moduli by the energy principles, one is confronted with the task of finding an admissible stress and strain field in the medium. To find the admissible stress and strain fields often the geometry of the phases in the composite materials is approximated by some simple geometrical model like sphere. Budiansky (105) attempted to evaluate the elastic moduli of a composite material with phase geometries being approximate spheres. The author used energy principles and self consistent approximation (106) to consider the interaction between the various phases. The results of Budiansky's work for an 'N' phase medium are given by

$$\sum_{i=1}^{N} \frac{v_i}{1 + \beta^* \left(\frac{G_i}{G^*} - 1\right)} = 1 \qquad .. (7.2)$$

$$\sum_{i=1}^{N} \frac{v_i}{1 + \alpha^* \left(\frac{K_i}{K^*} - 1\right)} = 1 \qquad .. (7.3)$$

$$\alpha^* = \frac{1 + \nu^*}{3(1 - \nu^*)}, \quad \beta^* = \frac{2}{15} \frac{(4-5\nu^*)}{(1 - \nu^*)} \qquad .. (7.4)$$

where v_i = volume fraction of the i th phase

G_i = shear modulus of the i th phase

ν^* = effective poisson's ratio of the composite material

G^* = effective shear modulus of the composite material

$K^*, \bar{K}_i$ = effective Bulk modulus of the composite material; Bulk modulus of the ith phase respectively

α^*, β^* = constants defined by Eq. 7.4

N = total number of phases

In the limiting case of holes or voids in a two phase mdedium the effective moduli, K^* and G^* are given by (105).

$$G^* = \frac{3(1 - 2v)}{3 - v} G \qquad .. (7.5)$$

$$K^* = \frac{4(1-2v)(1-v)}{v(3-v)} G \qquad .. (7.6)$$

where v = volume fraction of voids

G = shear modulus of the solid phase

Before concluding the discussion on the effective moduli of composite media, it may be mentioned that at present no sufficiently formal and simple approach is available to study the bulk behaviour of composite media. This may be noticed by the fact that the various bounds on effective moduli obtained by energy principles are too wide apart to be useful (101,104), if one of the phases happens to be voids or rigid. Of the various results available, Budiansky's (105) results are simple, eventhough approximate and consequently the results given by Eqns. 7.5 and 7.6 are used in subsequent development.

7.3 A THEORY OF FRACTURE BEHAVIOUR OF MATERIALS

In the following, it is proposed to develop a unified theory that relates the size effects on strength with other aspects of mechanical behaviour like ductile-brittle transition, size effect on stiffness and non-linearity in stress-strain behaviour. The basic concept of the theory is that with increased loading microcracks and flaws develop and progressively decrease the stiffness of the material as a consequence of progressive breakdown in internal structure. The extent

of microcracking with loading is related semi-empirically to the probability of failure at any stress level. The effective Young's modulus of the material in the presence of microcracks is obtained by using Budiansky's (105) result given by Eqns. 7.5 and 7.6 for the effective elastic moduli of a two phase composite material in the presence of voids. The effective Young's modulus, so derived is shown to be dependent on stress level and probability of failure and is shown to predict the nonlinearity in stress-strain behaviour, ductile-brittle transition, and stiffness as affected by specimen size. The detailed development of the theory is shown in Section 7.3.2.

7.3.1 Nature of Materials for which the theory is Applicable

The class of materials for which the theory is applicable is that in which permanent deformation due to microcracking takes place during subcritical crack growth. Concrete, rocks and fibre reinforced mortar are good examples of such materials, in which microcracks could be present in the material even before application of loading due to shrinkage and other effects. The microcracks progressively increase under loading and the microcracking forms an energy

dissipating mechanism other than surface energy. Such aforementioned materials which may be termed as 'concrete-like materials' display a nonlinear stress-strain behaviour and behave in a semi-ductile manner in contrast to brittle materials in which the onset of crack growth and failure are synonymous. In the following theory the feature of subcritical crack growth due to microcracking is incorporated and hence the theory is applicable only to concrete like materials, in contrast to ideally brittle or ideally ductile materials.

7.3.2 Development of the Theory

As has been already indicated, the basic concept of the theory is to treat concrete like materials as composite materials with one of the phases being voids due to microcracking. It is essential that the microcracks be given due recognition since they significantly influence the overall or macroscopic behaviour of the material. The microcracks could be present even under no load due to differential shrinkage and consequent bond failures at the aggregate-mortar interface, or could develop during loading. In any case the microcracks propagate further with increase in applied load and influence the overall behaviour

and strength. The size effect on strength i.e., the decrease in strength and associated scatter exist in concrete as has been already discussed in the proceeding chapters. To keep the generality of the study, it is assumed that the strength distribution function characterizing the probability of failure $\Phi^*(\sigma)$, for a stress level less than or equal to σ is given by

$$\Phi^*(\sigma) = 1 - \exp\{-V\,\bar{f}(\sigma)\} \qquad .. (7.7)$$

where $\Phi^*(\sigma)$ = probability of failure when the stress lies any where between 0 and σ

V = specimen size (volume)

$\bar{f}(\sigma)$ = is any function of σ that increases with σ and is such that it satisfies suitable conditions to make $\Phi^*(\sigma)$, a distribution function ($\bar{f}(\sigma)$ could be, for example $\left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m$ as in Weibull's distribution).

It could be observed from Eqn. 7.7 that the probability of failure $\Phi^*(\sigma)$ increases with increase in stress level for a given specimen size and the probability of failure increases with specimen size for a given stress level. The specific form of $\bar{f}(\sigma)$ is not necessary in the following theory.

A typical test on concrete in compression with measurements for linear, lateral and volumetric strains, Fig. 7.1 reveals that the stiffness of the

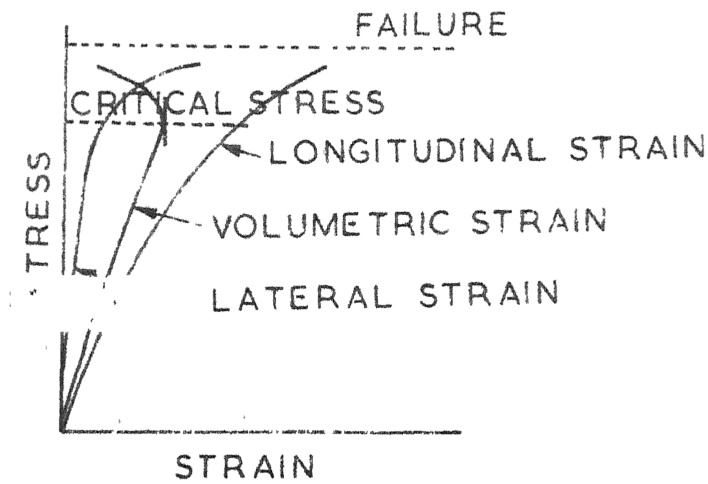


FIG.7·1 TYPICAL PLOT OF STRESS VS. LONGITUDINAL, LATERAL AND VOLUMETRIC STRAINS (107)

material goes on progressively decreasing with applied load and the stress - strain curve is nonlinear at each stage of loading. The nonlinearity in the stress-strain behaviour is attributable to the irreversible microcracking (89), the initiation of which becomes significantly manifest at applied loads of above 30% of the ultimate load. With further loading, the stiffness progressively decreases and when the applied load is about 70 to 90%,(107) of the ultimate, considerable breakdown in the internal structure takes place when the well developed microcracks start joining and form fissures so that at the ultimate stress level, the material system tends to form a mechanism. Further, an examination of the variation of volumetric strain with load indicates a progressive decrease in magnitude because the system is in compression, but with the onset of discontinuity in the material i.e., at about 70% of the ultimate load, the volume change reverses in nature. While many attempts have been made earlier to express the stress-strain relation of concrete by various analytical expressions (108), none of these are based on any formal theory. A formal approach to the problem is however, extremely difficult, but it appears that the same may be approached in a rational

manner with some suitable approximations and simplifications as discussed below.

In a study, wherein a theoretical evaluation of the stiffness of concrete as dependent on applied stress level is attempted, the following points should be incorporated. While a progressive decrease in stiffness of the material with increasing stress indicates that there is an increase in the volume of microcracks or voids in the material, the decrease in the gross volume change initially in the case of compression tests indicates, Fig. 7.1, that the solid phase decreases in volume with increased compressive stress. The macroscopically observed gross volume change is the net effect of increase in microcrack or void volume and that due to decrease in the solid phase. However at stress levels beyond about 70% of the ultimate load, the nature of variation of volumetric strain with load changes, as shown in Fig.7.1, indicating that at the loads nearing ultimate load, the increase in volume of microcracks or voids dominates the decrease in volume of the solid phase. In the case of tension tests, the phenomenon can be expected to be quite anologous to what is presently discussed, except that the gross volume increases monotonically upto failure.

As was mentioned earlier, Eqn. 7.7, the stress dependent risk of failure of specimen of size V can be given by a distribution function of the form,

$$\Phi^{*}(\sigma) = 1 - \exp\{-V\,\bar{\bar{f}}(\sigma)\} \qquad .. (7.8)$$

Since concrete-like materials display a progressive breakdown in internal structure with increased stress σ, it can be conceived that various elements constituting the material progressively fail and contribute to the microcrack or void volume (109). Consequently the materials do not obey the weakest link concept. To quantify the phenomenon of progressive breakdown of internal structure the cumulative probability of failure $\Phi^{*}(\sigma)$ can be visualized as follows. If the material of volume V can be divided into N elements and $n(\sigma)$ elements are likely to fail at an applied stress level σ, then the cumulative probability of failure $\Phi^{*}(\sigma)$ may be written as (109)

$$\Phi^{*}(\sigma) = 1 - \exp\{-V\,\bar{\bar{f}}(\sigma)\} = \frac{n(\sigma)}{N} \qquad . (7.9)$$

$$\text{or} \quad n(\sigma) = N\,[1 - \exp\{-V\,\bar{\bar{f}}(\sigma)\}] \qquad .. (7.10)$$

If $n(\sigma)$ elements get ruptured during loading upto a stress level σ, then the volume v_o occupied by the microcracks can be given by

$$v_o = k_o\, n(\sigma) = k_o N \quad [1 - \exp \quad \{-V\, \bar{f}(\sigma)\}] \quad .. (7.11)$$

where k_o = a constant that denotes the mean volume of a single microcrack.

The volume occupied by the n microcracks being v_o, the effective volume of microcracks or voids that contributes to the loss in stiffness can be given by

$$\bar{v} = k_1 \; v_o \qquad .. (7.12)$$

where $\bar{v}$ = effective volume of microcracks or voids that decrease the stiffness of the material

and k_1 = is a constant

It could be seen that Eqns. 7.9 through 7.12 correlate the probability of failure with effective void volume in the material. With this estimate of void volume, one may find the effective Young's modulus E^* in the presence of microcracks by using Budiansky's (105) results, given by Eqns. 7.2 and 7.3. Budiansky's values of effective bulk and shear moduli K^* and G^*, respectively can be rewritten as

$$G^* = \frac{3(1 - 2\bar{c})}{3 - \bar{c}} \quad G \qquad .. (7.13)$$

$$K^* = \frac{4(1 - 2\bar{c})(1-\bar{c})}{\bar{c}(3 - \bar{c})} \quad G \qquad .. (7.14)$$

where $\bar{c}$ = effective void or microcrack volume fraction $= \frac{\bar{v}}{V}$

G = the shear modulus of the solid phase

The effective Young's modulus E^* can be obtained from Eqns. 7.13 and 7.14 and is given by

$$E^* = \frac{9K^* G^*}{3K^* + G^*} = \frac{36(1-2\bar{c})(1-\bar{c})}{(3-\bar{c})\bar{c} + 4(1-\bar{c})(3-\bar{c})} \cdot G \quad .. (7.15)$$

or

$$E^* = \frac{36(1-2\bar{c})(1-\bar{c})}{(3-\bar{c})\bar{c} + 4(1-\bar{c})(3-\bar{c})} \cdot \frac{E}{2(1+\nu)} \quad .. (7.16)$$

$$= \frac{36(1-2\bar{c})(1-\bar{c})}{(3-\bar{c})\bar{c} + 4(1-\bar{c})(3-\bar{c})} \cdot \left(\frac{E}{2.4}\right) \quad .. (7.17)$$

where ν = 0.2, the Poission's ratio of the solid phase

and E = the Young's modulus of the solid phase

It is fortuitous that ν = 0.2 used in Eqn. 7.17, following the analysis of Budiansky, happens to be a good approximation for the Poisson's ratio of concrete during the initial stages of loading.

It can be seen from Eqn. 7.17 that as $\bar{c}$, the fractional volume of voids, increases E^* decreases and approaches zero when $\bar{c}$ equals a value 0.5, implying that E^* reduces to zero or failure of the material occurs, when the effective void volume reaches 50% of that of the total material. This is contrary to usual observations and is an outcome of the theory of Budiansky (105), as remaked by he himself.

Concrete, for example has void or microcrack volume at failure much less than 50% of the total volume. This is essentially the reason why the constant k_1 was introduced empirically in Eqn. 7.12. The constant k_1 provides for the necessary magnification of the actual void volume so that Budiansky's theory is made suitable and may be evaluated by experimentally measuring the fractional void volume at failure and setting

$$k_1 \frac{v_{failure}}{V} = 0.5 \qquad .. (7.18)$$

where $v_{failure}$ = void volume at failure

V = total volume

The effective Young's modulus E^* as given by Budiansky's theory, Eqn. 7.17 can be evaluated if $\bar{c}$, the fractional void volume is known. However, the fractional void volume $\frac{\bar{v}}{V}$ dependent on stress level σ, probability of failure $\Phi^*(\sigma)$ and specimen size V, is already given by Eqns. 7.11 and 7.12 and is given by

$$\bar{c} = \frac{\bar{v}}{V} = \frac{k_1 v_o}{V} = k_1 k_o N \quad [1- \exp\{-V \bar{f}(\sigma)\}] \quad .. (7.19)$$

The effective Young's modulus E^* is given by

$$E^* = \frac{36(1-2\bar{c})\ (1-\bar{c})}{(3-\bar{c})\bar{c} + 4(1-\bar{c})(3-\bar{c})} \left(\frac{E}{2.4}\right) = \frac{15\ E(1-2\bar{c})(1-\bar{c})}{(3-\bar{c})\bar{c}+4(1-\bar{c})(3-\bar{c})} \quad .. (7.20)$$

where $\bar{c}$ is given by Eqn. 7.19.

In the Eqn. 7.20 above, the effective Young's modulus E^* of the material is expressed in terms of $\bar{c}$, the fractional void or microcrack volume, Since $\bar{c}$ is a function of applied stress level σ and the specimen size V, the Young's modulus E^* is a function of specimen size and stress level. Consequently the theory developed yields a size and stress dependent value of Young's modulus E^* for materials in which microcracking is prevalent. Further implications of Eqns. 7.19 and 7.20 will be discussed in detail in the following chapter.

7.4 DISCUSSION

In this chapter an expression for the effective Young's modulus of a material in the presence of microcracking is developed using an estimate of the volume of microcracks as dependent on stress level and specimen size. The growth of microcracks is assumed to contribute to the effective void volume in the material. The material is considered as a two phase composite material in which one of the phases is voids and the related result of Budiansky (105) for effective Young's modulus is invoked in the theory. The choice of Budiansky's result is purely for its simplicity and the logical basis on which it is developed.

However, to compensate for the fact that the fractional void volume at failure is much less than that predicted by Budiansky's result, an additional constant k_1 was to be introduced in the theory which could be avoided if a more general and accurate result for effective Young's modulus of a composite material in the presence of voids is available. The implications of the theory developed in terms of the nonlinearity in stress-strain behaviour, size effect on stiffness and ductile-brittle transition will be discussed in detail in the following chapter.

CHAPTER EIGHT

THEORY OF SIZE EFFECTS ON MECHANICAL BEHAVIOUR OF CONCRETE-LIKE MATERIALS

8.1 NONLINEARITY IN STRESS-STRAIN BEHAVIOUR

Truesdell (66) remarks that the various theories of nonlinear constitutive laws can be classified to fall into two groups depending on the basic approaches. In either of the approaches, a description of the quality of the material and a method of quantifying it are attempted. In the classical theory of materials viz, elasticity, the quality of material (perfect elasticity) is quantified in the simplest manner by linearity. Starting from this basis, in the first approach, additional qualities to the material are assigned while the quantification is still linear as in plasticity theory; in the second approach, the quality, namely elasticity is preserved but the quantifying measure is changed as in the theories of finite elasticity. The suitability of the two approaches, however is governed by the material for which it is applied. Viewing in this light, the following discussion of the results of the preceding chapter can be observed to fall into the first category in the broad classification of Truesdell.

8.1.1 Nonlinearity due to Progressive Breakdown in Internal Structure

The Eqns. 7.19 and 7.20 relating the effective modulus E^* and effective fractional void volume $\bar{c}$, derived in the last chapter are as follows:

$$\bar{c} = k_1 \, k_o N \, \Phi^* (\sigma) = k_1 \, k_o \, N \, [1- \exp \, \{-V \, \bar{f} \, (\sigma)\}] \quad .. \, (8.1)$$

$$E^* = \frac{15E \, (1-2\bar{c}) \, (1-\bar{c})}{(3-\bar{c})\bar{c}+4(1-\bar{c})(3-\bar{c})} \quad .. \, (8.2)$$

The stress-strain law of the material as described by the above Eqns. 8.1 and 8.2 is essentially non-linear, since the elastic modulus E^* contains σ through $\bar{c}$. It is obvious that the nonlinearity is an outcome of the fact that the microcracks which contribute to the void volume, increase with stress σ. From Eqn. 8.1, it can be seen that as σ increases, $\bar{c}$ the effective fractional void volume increases and as $\bar{c}$ increases E^* decreases, as given by Eqn. 8.2. When $\bar{c}$ tends to a value 0.5, E^* reduces to zero. The variations of $\bar{c}$ and E^* with σ and the stress-strain relationship according to Eqn. 8.2 are shown schematically in Figs. 8.1 to 8.3. The slope of the stress-strain curve given in Fig. 8.3 denotes E^*, the effective Young's modulus. The predictions are

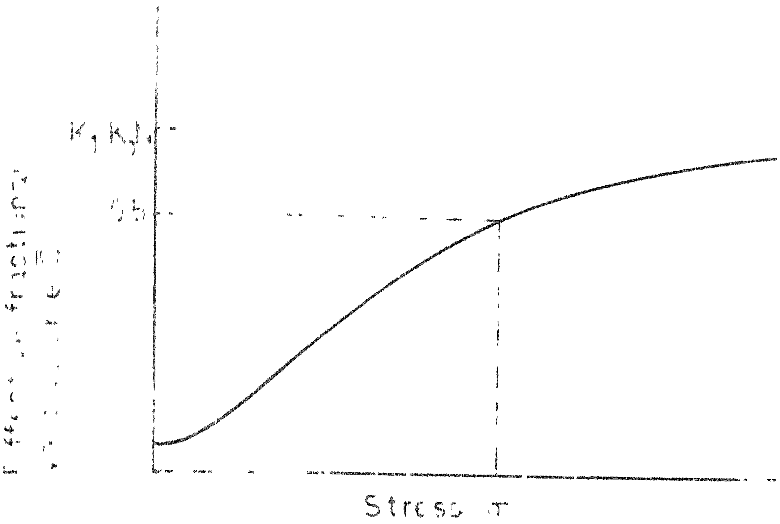


FIG. 8.1 VARIATION OF EFFECTIVE FRACTIONAL VOID VOLUME $\bar{C}$, WITH STRESS σ

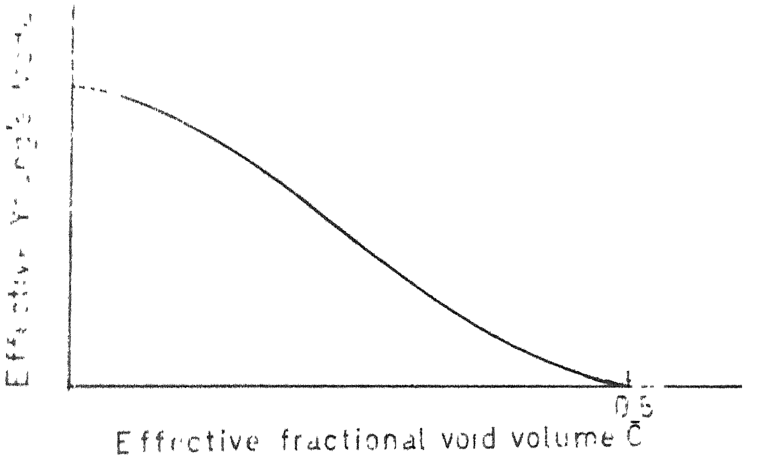


FIG. 8.2 VARIATION OF EFFECTIVE YOUNG'S MODULUS E^* WITH EFFECTIVE FRACTIONAL VOID VOLUME $\bar{C}$

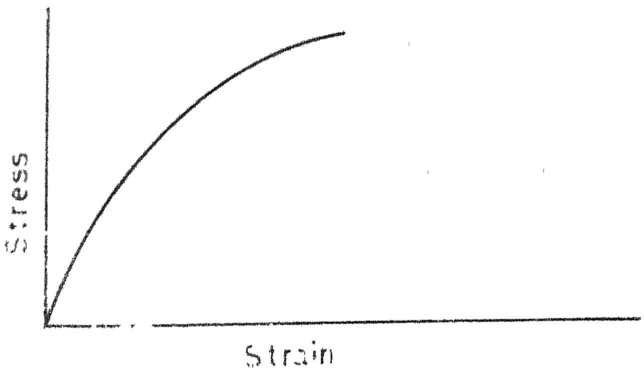


FIG. 8.3 NON LINEAR STRESS-STRAIN BEHAVIOUR

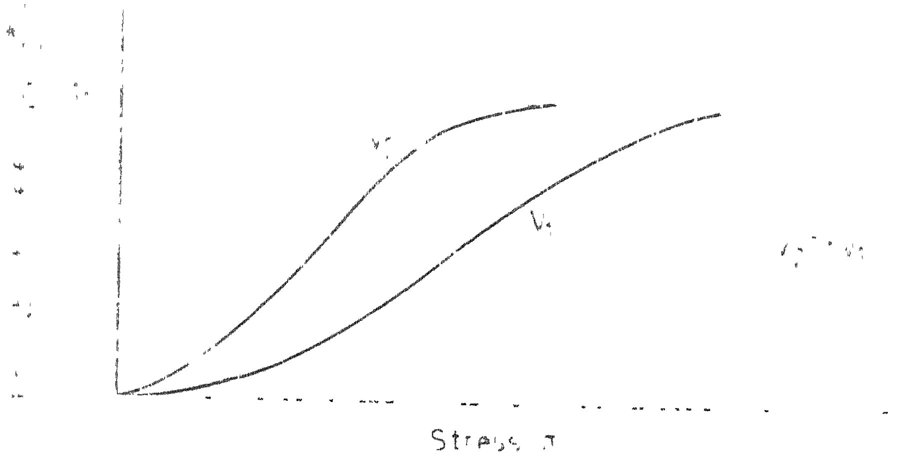


FIG. 8·4 VARIATION OF $\bar{p}^*(\sigma)$ WITH σ AND V

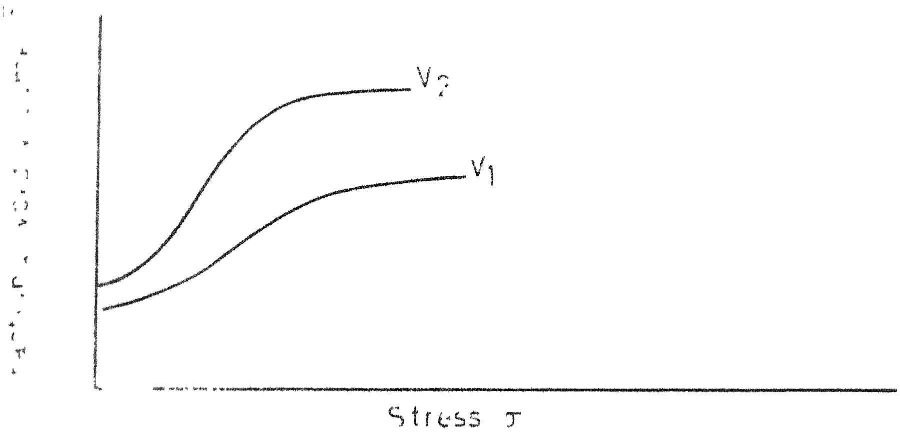


FIG. 8·5 VARIATION OF $\bar{C}$ WITH σ (EQN. 8 1)

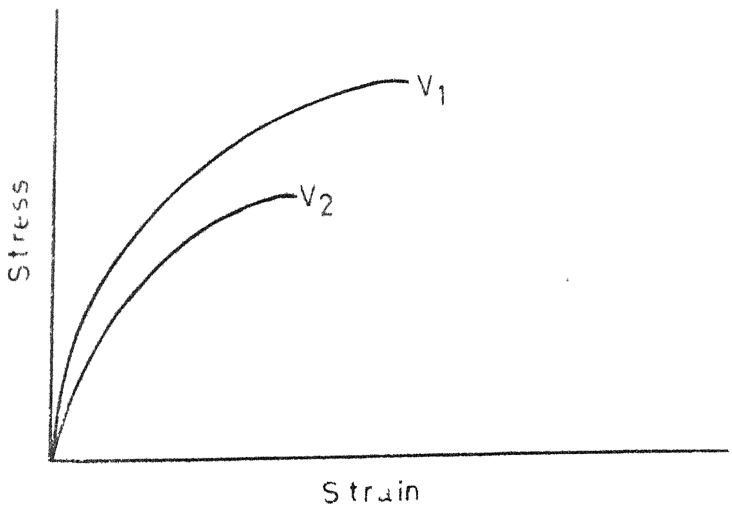


FIG. 8·6 SIZE DEPENDENT STRESS STRAIN RELATION

$$\Phi_1^*(\sigma) = 1 - \exp\{-V_1\,\bar{f}(\sigma)\} \qquad ..(8.3)$$

$$\Phi_2^*(\sigma) = 1 - \exp\{-V_2\,\bar{f}(\sigma)\} \qquad ..(8.4)$$

$$\Phi_2^*(\sigma) > \Phi_1^*(\sigma) \qquad ..(8.5)$$

where $\Phi_1^*(\sigma)$ = the probability of failure for any stress level less than or equal to σ, of specimen of size V_1

$\Phi_2^*(\sigma)$ = the probability of failure for any stress level less than or equal to σ, of speciment of size V_2

Considering that the number of macroscopic volume elements in the material per unit volume to be given by n^*, the total number of volume elements N in Eqn. 8.1 is given by

$N_1 = n^* V_1$ in the case of specimen of size V_1 ..(8.6)

$N_2 = n^* V_2$ in the case of specimen of size V_2 ..(8.7)

where

N_1 = total number of macroscopic volume elements in the specimen of size V_1

N_2 = total number of macroscopic volume elements in the specimen of size V_2

The void volumes at any stress level σ in either of the two specimens of sizes V_1 and V_2 can be obtained relating the probabilities of failure $\Phi_1^*(\sigma)$ and $\Phi_2^*(\sigma)$ to the number of ruptured elements in the two specimens, as follows:

$$\Phi_1^*(\sigma) = \frac{n_1(\sigma)}{n^* V_1} \quad .. (8.8)$$

and $$\Phi_2^*(\sigma) = \frac{n_2(\sigma)}{n^* V_2} \quad .. (8.9)$$

where n^* = number of macrocropic volume elements per unit volume

$n_1(\sigma)$ = number of elements that get ruptured at a stress level σ in the specimen of size V_1

$n_2(\sigma)$ = number of elements that get ruptured at a stress level σ in the specimen of size V_2

Since the ruptured elements are considered to contribute to the effective microcrack or void volume, as discussed in the previous chapter

$$\begin{aligned}\bar{v}_1 &= k_1\, k_o\, n_1(\sigma) \\ &= k_1\, k_o\, n^*\, V_1\, \Phi_1^*(\sigma) \\ &= k_1\, k_o\, n^*\, V_1\, [1 - \exp\{-V_1\, \bar{f}(\sigma)\}] \quad .. (8.10)\end{aligned}$$

and similarly

$$\bar{v}_2 = k_1\, k_o\, n^*\, V_2\, [1 - \exp\{-V_2\, \bar{f}(\sigma)\}] \quad .. (8.11)$$

where $\bar{v}_1$ = effective volume of microcracks or voids in specimen of size V_1

$\bar{v}_2$ = effective volume of microcracks or voids in specimen of size V_2

k_o = constant denoting the mean volume of a single microcrack (defined in the previous chapter)

and k_1 = semiempirical constant defined by Eqn. 7.18 in the previous chapter

From Eqns. 8.10 and 8.11, the fractional effective void volume in either of the specimens can be written as

$$\bar{c}_1 = \frac{\bar{v}_1}{V_1} = k_1\ k_o\ n^* \quad [1- \exp \quad \{-V_1\ \bar{f}(\sigma)\}] \quad .. (8.12)$$

and

$$\bar{c}_2 = \frac{\bar{v}_2}{V_2} = k_1\ k_o\ n^* \quad [1- \exp \quad \{-V_2\ \bar{f}(\sigma)\}] \quad .. (8.13)$$

From Eqns. 8.12 and 8.13, for a given σ, since $V_2 > V_1$

$$\bar{c}_2 > \bar{c}_1 \quad .. (8.14)$$

The Eqn. 8.14 implies that, for a given stress level σ, the effective fractional void volume is more for a larger specimen. From Eqns. 8.2 and 8.14, it can be observed that the effective Young's modulus E^*, is less for the larger specimen, at a given stress level σ. If E^* denotes the slope of the stress-strain curve, then the stress-strain curves corresponding to specimens of different sizes would be, according to Eqns. 8.2 and 8.14, as shown schematically in Fig. 8.6. The Fig. 8.6 shows the predicted size effect on stress-strain behaviour. The mean failure stress is denoted by E^* reaching a value zero. The larger

specimen has a lower failure stress as E^* approaches zero more rapidly with σ according to Eqns. 8.2 and 8.13.

8.2 SIZE EFFECT ON STIFFNESS

Fig. 8.5, which shows schematically the size effect on stress-strain behaviour also shows the size effect on stiffness. From the discussion in the preceding section and the development of Eqns. 8.2 to 8.14, it follows that, at any given stress level, the larger specimen has a greater effective void volume and hence a lower effective Young's modulus E^*, implying that larger specimens are less stiff as compared to smaller specimens.

8.3 SIZE EFFECT ON DUCTILE-BRITTLE TRANSITION

The effects of temperature and state of stress on ductility of materials are well known. However, the effect of specimen size on ductility is less known and is essentially brought to light by Glucklich (58, 61,62). In the following a brief account of the observations and theory of Glucklich is given. The predictions of the present theory with respect to size effect on ductile-brittle transition are compared with those of Glucklich in Section 8.3.2.

8.3.1 Earlier view Point of Strain Energy Size Effect

Glucklich (62) examined the load deflection profiles of notched beams of different sizes and the stress-strain curves as obtained from notched beams of gypsum mortar loaded by a spring. In addition, Glucklich studied the effect of loading with a spring in series in compression and tension on specimens of Gypsum mortar. Glucklich noticed that the load deflection profiles of notched beams depicted that greater energy was absorbed before failure in smaller specimens. The presence of a spring in series with a notched beam had the similar effect viz, specimens with no spring displayed greater ductility and strength than those with a spring. The observations indicated that the presence of a spring in series with a specimen under load is essentially same as having a larger specimen with reference to strength and ductility. Larger specimens showed less strength and ductility as compared to smaller specimens. So also, comparing the behaviour of specimens with identical size but with and without a spring in series, the specimen tested with a spring in series showed less strength and lower ductility. The above phenomena are commonly termed as Strain Energy Size Effect, by Glucklich (58,62). The theory advanced by

Glucklich to interpret these phenomena is essentially based on the total energy of the system under loading and balance of energy during crack propagation. The theory of Glucklich, briefly, is as follows. In an ideal Griffith material, the surface energy is the only mechanism of energy dissipation and a single potential crack and its propagation is considered. Also the ideal brittle material is homogeneous with a uniform energy demand during crack propagation. Consequently an equilibrium exists between crack length and external load during stable crack propagation. As the load and crack length increase, a stage is reached when the strain energy release rate exceeds that of energy absorption due to surface energy and spontaneous fracture occurs. However in materials like concrete and many other real materials, the energy balance is not as described above because of the presence of microcracking and the presence of harder phases. Whenever a propagating crack encounters a harder phase greater energy is required to force the crack further. Also when once the crack encounters a weaker zone, the excess energy, if present works against the uncracked material. Consequently in such materials the total energy of the system which acts as a reservoir determines the mode of crack propagation

and failure. If a system has higher energy level due to the presence of a spring in series or a larger specimen, the excess energy drives the crack through the harder phases and the following kinetic energy advances fracture, favouring the conditions for unstable crack propagation. Because the deformation is smaller during unstable crack propagation, the presence of the energy reservoir due to spring or a larger specimen size brings down the ductility of the system also. Based on the above concepts, Glucklich proposed a transition size anologous to transition temperature, which is schematically shown in Fig. 8.7.

8.3.2 Evaluation of the Present Theory in the Light of the Concept of Strain Energy Size Effect

Taking total area under the stress-strain curve upto to the failure stress, i.e. the energy absorbed before rupture as a measure of ductility it can be noted that the size dependent stress-strain curve shown in Fig. 8.6 and discussed in Section 8.1.2, explains that larger specimens display less ductility as compared to smaller specimens. This feature is schematically shown in Fig. 8.8. It should however be noted that the theory is applicable upto maximum or failure stress, but not beyond, for example, in the strain softening range. The view point advanced in

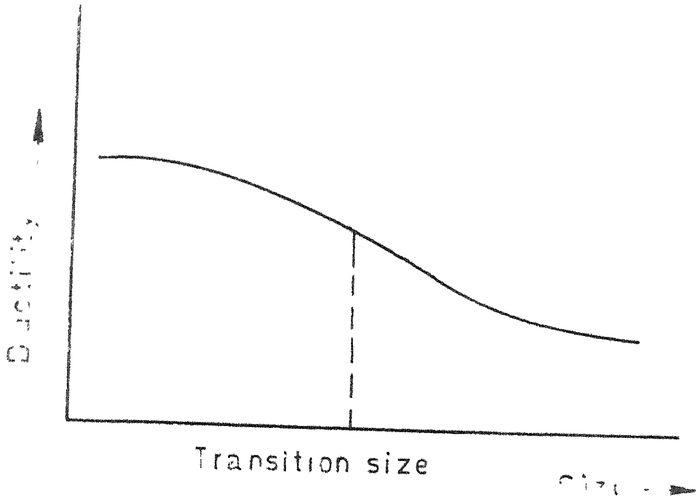


FIG 8·7 TRANSITION SIZE, AFTER GLUCKLICH (58)

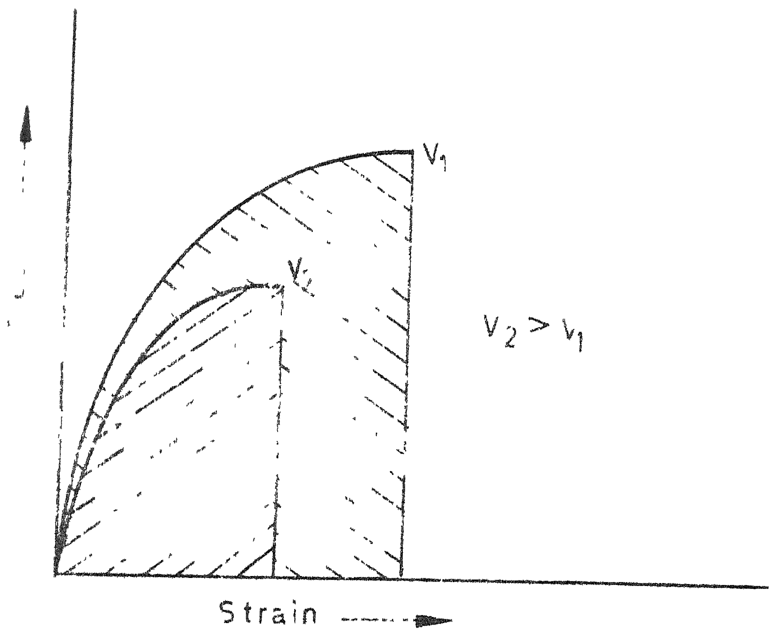


FIG 8·8 ENERGY ABSORBED BEFORE RUPTURE IN SPECIMENS OF DIFFERENT SIZES

the present theory, namely, that the microcracks contribute to void growth which lowers the material stiffness, is different from that of Glucklich wherein the total energy content of the system and the energy balance, determines the fracture mode. In the present theory, size effect on strength, stiffness and nonlinearity in material behaviour are inherent in the material that displays size effect on ductile-brittle transition. While the theory of Glucklich emphasises on the basic mechanism of fracture in the materials, the present theory examines the behaviour essentially on a macroscopic level and relies on the quantification of fracture process on the average, incorporating various known characteristics of deformation of the material like microcracking and volume change within the framework of the theory of composite materials. Both theories being qualitative at present, a detailed comparison is not possible. A more specific comparison needs further study of both the theories as applied to specific materials.

8.4 EXPERIMENTAL EVIDENCE IN SUPPORT OF THE THEORY

Systematic experimental investigations into the size effects on strength, stiffness and stress-strain curve together on any material are not reported in the literature. A preliminary experimental investigation

carried out on fibre reinforced cement mortar specimens, to examine the predictions of the theory is described below. Fibre reinforced cement mortar is a semiductile material which has stable crack propagation and admits a progressive break down in internal structure under increased loads due to microcracking in mortar as well as the bond rupture at fiber - mortar interface. Since the strains are relatively larger, the experimental measurements are more accurate as compared to concrete and similar materials in tension.

8.4.1 Details of Tests on Steel Fiber Reinforced Cement Mortar Briquettes

Four series of tests on fibre-reinforced cement mortar briquettes in tension were carried out with briquette thickness as the variable. The cement mortar prepared was of 1:3 mix consisting of sand of fineness modulus 2.1 and standard Portland Cement. The embedded steel fibres were of 18 gage and yield strength 38 Kg. In series 1,2 the fibre content was 8% by volume while in the series 3 and 4 it was 16% by volume. In each of the series the briquettes were prepared and tested under identical conditions. The briquette sizes are shown in Figs. 8.9 and 8.10. Table 8.1 gives the details and number of specimens

TABLE 8.1: DETAILS OF TEST SPECIMENS OF STEEL FIBER REINFORCED CEMENT MORTAR

Series	Fibre volume %	Briquette thickness inches	Number of specimens tested
1	8	0.5	3
2	8	0.75	3
3	16	0.5	3
4	16	0.75	3

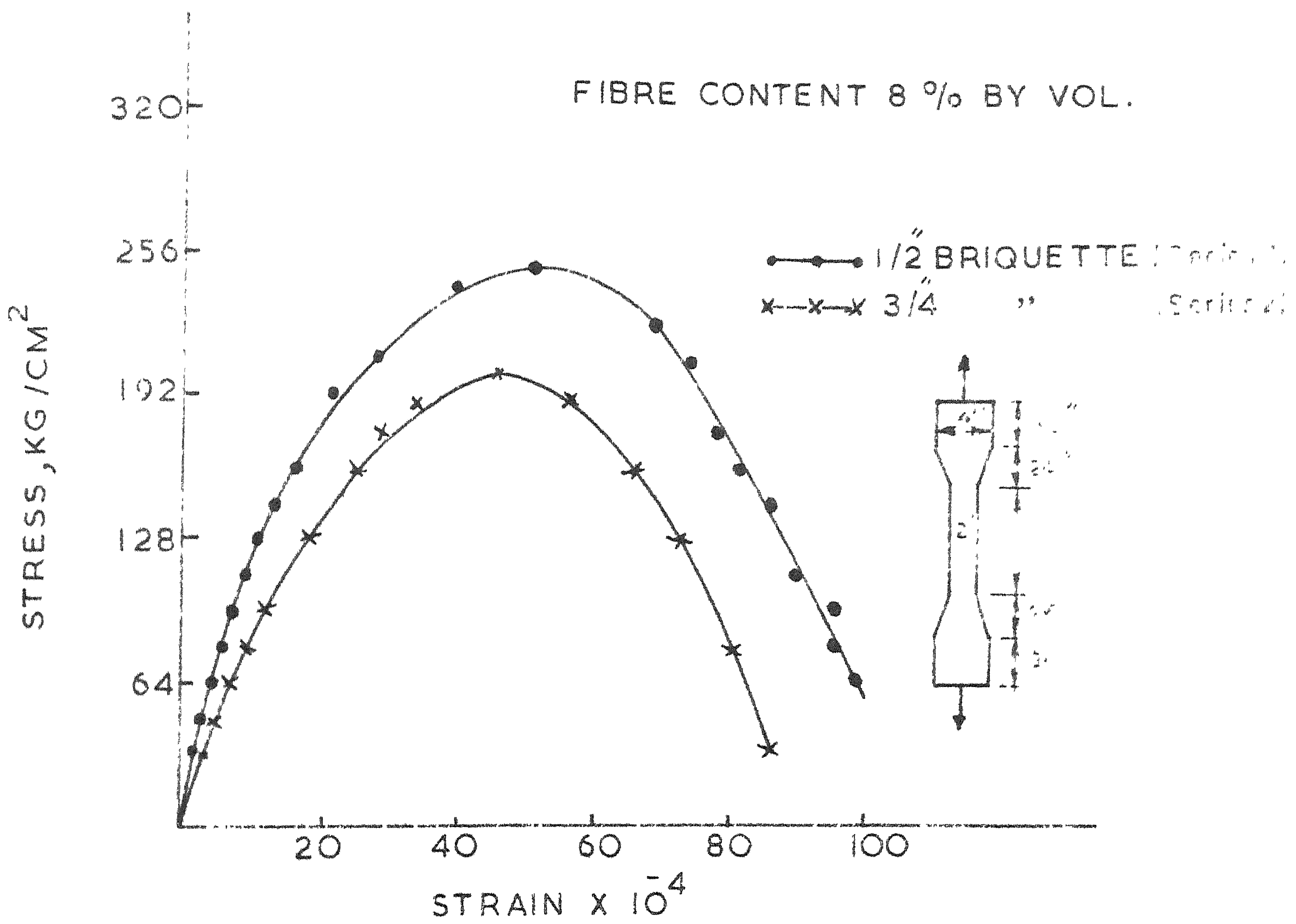


FIG.89 STRESS-STRAIN CURVES OF FIBRE REINFORCED MORTAR WITH SPECIMENS OF DIFFERENT SIZES

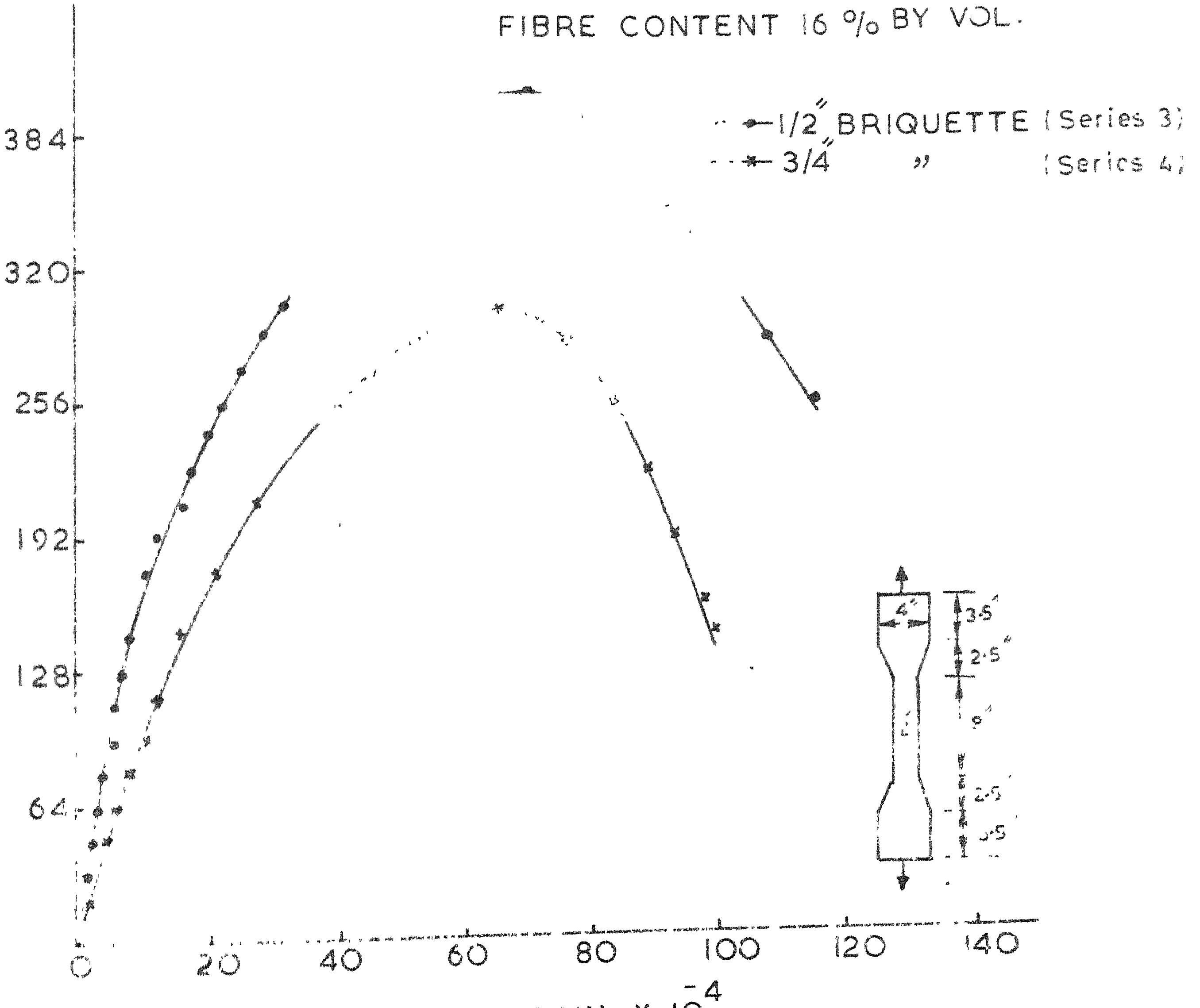


FIG. 8·3 STRESS STRAIN CURVES OF FIBRE REINFORCED MORTAR WITH SPECIMENS OF DIFFERENT SIZES

tested in each of the series.

8.4.2 Test Results

The stress-strain curves as obtained from tests with briquette thickness as the variable are shown in Figs. 8.9 and 8.10. The Figs. 8.9 and 8.10 clearly indicate that a size effect on stress-strain behaviour and strength exists. Also smaller specimens have higher strength, larger ductility and greater stiff - ness. The strains plotted in the Figs. 8.9 and 8.10 are the average values of the three specimens tested in each case and consequently indicate the average behaviour.

8.5 COMPARISON OF EXPERIMENTAL OBSERVATIONS AND THEORETICAL PREDICTIONS

The theory of size effects on materials proposed in the thesis predicted smaller stiffness and less ductility associated with larger specimens. The predictions are qualitative and are in agreement with the experimental observations on fibre-reinforced cement mortar briquettes in tension as shown in Fig. 8.9 and 8.10. The theory implied that the failure of material and the effective Young's modulus reaching zero are synonymous. Further,the descending portion of the stress strain curve, i.e., the strain soften-ing portion is not included in the theory discussed

and the predictions are valid only upto the maximum stress level. However, the Figs. 8.9 and 8.10 indicate higher ductility of smaller specimens, even by considering the area under complete stress strain curve as a measure of ductility.

8.6 DISCUSSION

The expression for effective Young's modulus of material as given by Eqns. 8.1 and 8.2, predicted adequately the size effect on stiffness, stress-strain behaviour and ductility at failure as evidenced by the experimental results discussed in Section 8.4. The predictions were essentially qualitative since the constants k_1, k_0 and N are to be fixed suitable to any particular material. The constant k_0 denotes the mean volume of a typical microcrack. k_1 is a constant that was empirically introduced to transform the microcrack volume to the effective void volume at failure as was discussed in Section 7.3.2, Eqns. 7.12 and 7.18, while N denotes the number of elements into which the material is devided in the development of the theory and is only of analytical significance. These constants may be considered as parameters characterizing the property of the material for which the theory is applied. While at present no specific suggestion can be offered as to the experimental

determination of the constants, the quantity 'k_1 k_0 N' may be taken together to denote a single constant of the material and the same may be quantified in any specific application. The qualitative implications of the theory namely, the greater stiffness, greater ductility associated with smaller specimens are however valid for any positive practical values of the constant k_1 k_0 N.

The theory is not very rigorous in its formulation and is developed by fitting several physical concepts within a logical frame work. Concrete-like materials are considered as two phase composite materials with one of the phases as voids. The microcracks in concrete like materials are related to the voids in the composite material. It is not implied that the correspondence between microcracks and voids is by any means exact; nevertheless, the correspondence is invoked only to quantify the stress dependence of Young's modulus in such materials. The probability of failure which is dependent on stress level and specimen size, assumed to be known, is related to the microcrack volume and hence to the effective Young's modulus of the material. The theory so developed adequately explained the size effect on stiffness, stress-strain behaviour and ductile-brittle

transition in concrete like materials. The transition from ductile to brittle as the specimen size increases is essentially gradual. The predictions of the theory are observed to be qualitatively in agreement with observations in tests on fibre reinforced cement mortar briquettes. Further exhaustive tests are essential, especially with measurements on volume changes and extent of microcracking at various stress levels, for a quantitative evaluation of the theory.

CHAPTER NINE

MATERIAL CHOICE FOR FRACTURE RESISTANT DESIGN OF PRESSURE VESSELS

9.1 INTRODUCTION

In the previous chapters, statistical aspects of size effects on strength and fracture behaviour have been investigated. In this chapter, a study is made on an aspect of fracture resistant design as an illustrative example of use of fracture mechanics in engineering design. The presence of crack like defects in structures like pressure vessels, pipe lines, ship hulls etc., can lead to catastrophic failures, if not gourded against in the design process. The conventional design process based on specified nominal stress level is inadequate unless the absence of cracks in structures is guaranteed during fabrication or subsequent service life. However, cracks in structures can be initiated and propagated during the fabrication as well as during their service life. In the presence of a crack the integrity of a structure is impaired depending on the severity of the crack with reference to its location, size and shape. Application of fracture mechanics concepts in structural design enables the engineer to assess the tolerable cracks

in a structure working at a particular stress level or, conversely, the allowable stress or load level in the presence of a crack of known characteristics. In the actual design problem where there is a likelihood of brittle fracture occuring, the fracture toughness and crack propagation characteristics of a material to be used are to be considered in the design process in addition to the other properties of the material like unit cost, density etc. It is attempted in this chapter to develop a design procedure for fracture resistant design as applied to pressure vessels. The emphasis in the design procedure is on the optimal material choice from an available set of materials with known properties. In the following, the need for fracture resistant design in pressure vessels is briefly discussed. Subsequently, a specific problem of pressure vessel design in the presence of a part-through crack, with optimal material choice is discussed in detail.

9.2 DESIGN CRITERIA FOR PRESSURE VESSELS

Several failures of welded pressure vessels, oil and gasoline storage tanks and pipe lines (121) warrant the need for fracture resistant design of such structures. While it is known that crack like defects originated either due to defective welding or during service life

are known to initiate brittle failures, the exact origin of the strength impairing defects is not clearly known. However, the presence of cracks in structures can initiate fracture even at service stresses much lower than the yield stress of the materials. The conditions for fracture in structures like pressure vessels and pipe lines are more favourable for brittle fracture occurance where the inherent ductility of the material is impaired due to various reasons. In the case of pressure vessels operating in a nuclear atmosphere, the neutron bombardment on the vessel material improves the yield and ultimate strengths of the material but significantly lowers the ductility and impact strength (122). Also, in the manufacture of pressure vessels, plates which are initially flat are bent into desired shape resulting in the cold forming of the material and as a consequence the material is prestrained which raises the yield point but reduces the ductility. In such cases the properties of the material prior to deformation need not be representative of those in the actual structure and this is more severe for fracture resistance (122). Pressure vessels under external shock loads and internal pressure cycles must have adequate ductility to operate economically at service temperatures. In a design attempted to realize such an end, the material

choice for the pressure vessel must be made with adequate fracture resistance.

9.3 FRACTURE RESISTANT DESIGN OF PRESSURE VESSELS

9.3.1 Problem of Fracture Resistant Design

While designing structures applying the concepts of fracture mechanics, the existence of cracks in structures is inherently admitted. In the design process one assesses the allowable stress level and the associated service life in the presence of flaws. The most important factor in the design is an estimate of the initial crack size, location and shape. Unfortunately the initial cracks often are very small, that they cannot be directly measured easily or could be observed by any nondestructive testing techniques. Consequently an estimate of the initial crack size and shape could only be made by an intelligent guess based on experience, observation of the previous designs and some knowledge of the fabrication and welding procedures adopted. Even though no formal procedure exists for initial crack size estimation, proof testing can be of some help in many cases (121). The method of assessing initial crack size using proof testing will be discussed in the problem attempted subsequently in this chapter. In designing pressure vessels for

fracture resistance, two approaches can be followed (i) Fail safe design (ii) Safe life design. The most critical cracks in pressure vessels are those which initially penetrate through the vessel wall only partly, called "part-through cracks". Those cracks which are through the wall thickness fully are usually detected by a leakage and could atleast be noticed. In contrast, the part-through cracks even though of such size initially that they are harmless, could grow during service life and may finally be of such size that catastrophic fracture occurs. In the fail safe design, the material and the wall thickness are so chosen as to ensure that the crack cannot catastrophically propagate unless it grows to such size that it penetrates completely through the wall thickness so that a leakage is indicated before fracture. In the safe life design, for a chosen material and wall thickness, the service life is estimated before fracture occurs and the structure is put out of service before the end of the design or actual life. The service could be resumed if the grown crack is of such size that it could be located and accurately measured and if it is found that further life can be ensured before rupture. It is the aim of the present chapter to suggest a design method so that the material choice could be optimally made for the design of pressure vessels by either of

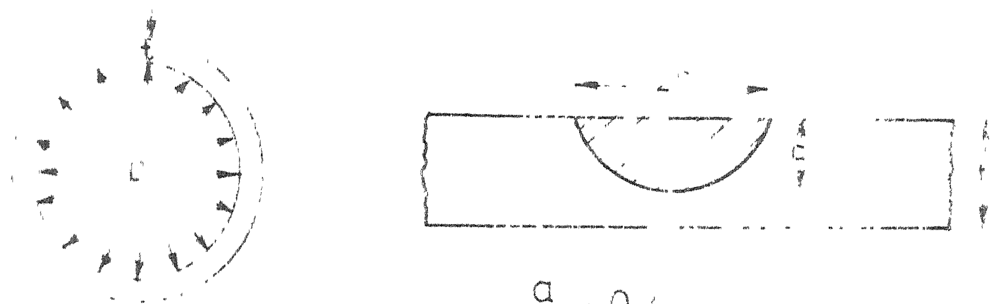
the approaches viz, the fail safe and safe life designs. The method is illustrated by a specific example in the following sections.

9.3.2 <u>An Example of the Design of a Pressure Vessel in the Presence of a part-through Crack</u>

A circular cylindrical pressure vessel of length 1000 inches with an internal pressure of 2000 psi. is to be designed, i.e., the material and thickness of the shell are to be chosen. Both safe life and fail safe designs are to be attempted and it is given that the part-through crack is semielliptical in shape and the ratio of flaw depth to the major axis is 0.4 as shown in Fig. 9.1. The nominal working stress should not exceed sixty percent of the yield stress of the material used. It is required to find an optimal material

(i) When the vessel diameter is fixed at 50" (it turns out to be safe life design as will be shown below)

(ii) When the vessel diameter can be varied but warning before fracture is given, in the form of a leakage i.e., the crack grows so that the flaw depth at failure is atleast as much as the vessel wall thickness. In either of the two cases, the design should be optimal with respect to material cost, service life and the weight of the structure. The utility value of the design is considered as the net return from the design, viz, the



$\frac{a}{2c} = 0.4$

Internal pressure p = 2[illegible]

[illegible] To choose the optimal material when [illegible] (Safe [illegible]

[illegible] To choose the optimal [illegible] and the thickness [illegible] warning before fracture is given (Fail safe [illegible])

FIG. [illegible] EXAMPLE OF [illegible] VESSEL.

return from service life less that incurred due to initial cost and the maintenance cost dependent on the weight of the structure. The material that maximizes the utility is the optimal material. The utility U_R, the net monetary return is taken as given below:

$$U_R = K_1 N_S - K_2 N_S w - K_3 C \quad .. (9.1)$$

U_R = net monetary return in Rupees

N_S = service life in cycles

w = weight of the vessel in lbs/inch length

C = material cost Rs/inch length

K_1^* = constant denoting the return from one cycle of service = 0.5, in this example

K_2^* = constant denoting the cost incurred per cycle of service depending on weight = 0.01 in this example

K_3^* = denoting the total length of the vessel = 1000 in. in this example

The return from service life and cost incurred as assumed in the present problem are shown in Figs. 9.2 and 9.3. It might be noted that in its totality, the problem of optimal design with material choice is a multifactor optimization problem (4, 123), wherein an optimum with reference to different criteria viz, weight, cost and service life is sought. It is necessary that a suitable trade off be made, so that the value of the design or

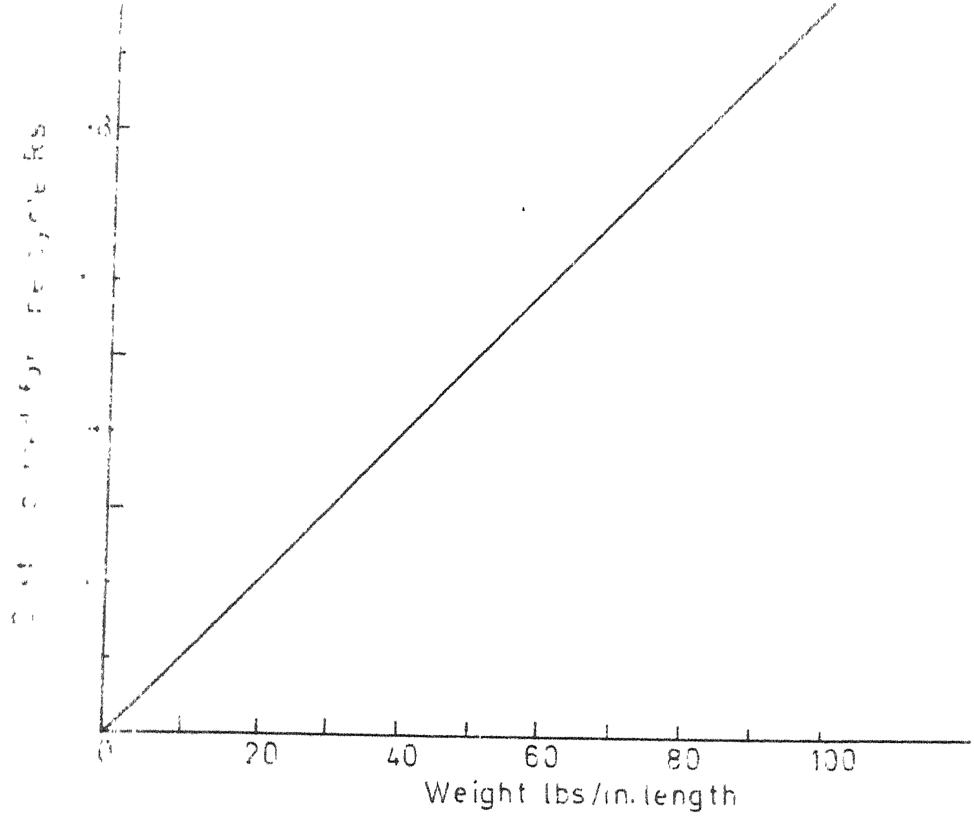


FIG. 9·2 COST INCURRED DUE TO WEIGHT

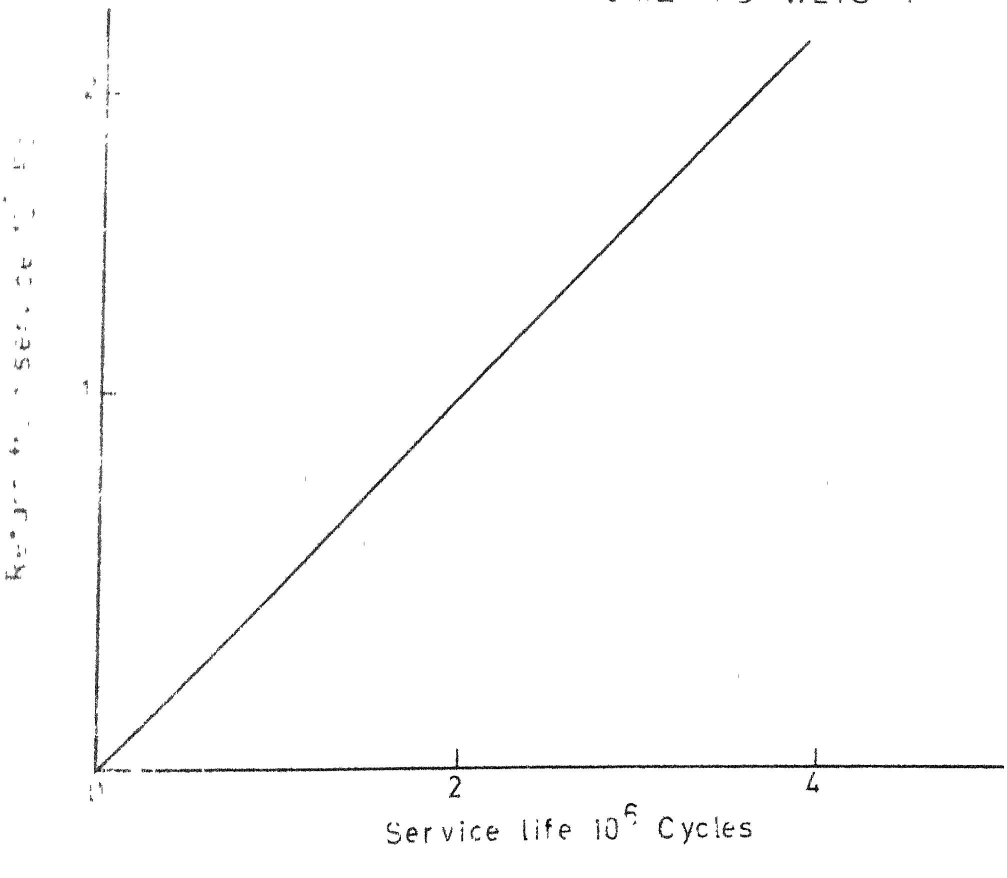


FIG. 9·3 RETURN FROM SERVICE LIFE

utility is maximized. The definition of utility as is given in the present problem is not necessarily the most general one and any other alternate definition could as well be given depending on the designer's knowledge of the outcomes of various design decisions. An alternate form of utility model with trade off between various design criteria is attempted by Blake (5) and Murthy (6); however further work is necessary to apply such models to practical design problems. In the present problem the utility as defined in Eqn. 9.1 is used as a measure of optimality in the material choice.

9.3.3 Materials Considered and their Properties

Six "candidate" materials are considered (for deciding the best one) and their properties are given in Table 9.1. The materials are designated by the corresponding serial number in Table 9.1, in the subsequent discussion. Because of the nonavailability of various properties for the same material, some of the properties have been assumed and are indicated by an asterisk in the table. The rate of propagation of a part-through crack which is of interest in estimating the service life does not appear to have been studied for any material. Tetelman and McEvily (128) reported the rate of crack propagation in steel specimens on

TABLE 9.1: PROPERTIES OF MATERIALS CONSIDERED FOR PRESSURE VESSEL DESIGN

Sl. No.	Material	Yield strength ksi	Ultimate strength ksi	Fracture toughness ksi $\sqrt{in}$	Density* lb/cu"	Youngs modulus* 10^6 psi	Cost Rs. per lb*	Strain at necking instability* %	Crack propagation factor β
	18Ni-Co-Mo steels (126) grade								
1	300-1	286	345*	51.75	0.283	30	3	11	1888
2	250	259	345*	68.4	0.283	30	4	11	1865
3	300-2	242	345*	84.5	0.283	30	5	11	1850
	Aluminium Alloys (125)								
4	2014-T651	64.3	69.7	28.2	0.107	10.6	2	11.6	128
5	2219-T87	57.6	69.3	27.9	0.107	10.6	2	10.5	246
6	7075-T6	73.2	80.7	25.8	0.107	10.6	2	11.5	157.5

* Assumed

7075-T6 Aluminium alloy and AM 355 CRT steel. The rate of crack propagation is expressed as a function of modified stress intensity factor K_m denoted by

$$K_m = \frac{\sigma_g \sqrt{c}}{\{(\frac{\sigma_Y + \sigma_U}{2})\ \varepsilon_u \sigma_U^2\ E\}^{1/4}} \quad \text{in}^{1/2} \times 10^2 \quad .. (9.2)$$

where
K_m = modified stress intensity factor
σ_g = nominal stress applied
c = initial crack length
σ_Y = yield stress
σ_U = ultimate stress
ε_u = strain at necking instability
E = Young's modulus

The relationship between K_m and rate of crack propagation as given by Tetelman and McEvily is reproduced in Fig. 9.4 and is assumed valid for the high strength steels and Aluminium alloys considered in Table 9.1 in the presence of a part-through crack. The quantity β for various materials denoted as crack propagation factor in Table 9.1 is the expression in the denominator of Eqn. 9.2 and is given by

$$\beta = \{(\frac{\sigma_Y + \sigma_U}{2})\ \varepsilon_u\ \sigma_U^2\ E\}^{1/4} \quad .. (9.3)$$

where β = crack propagation factor

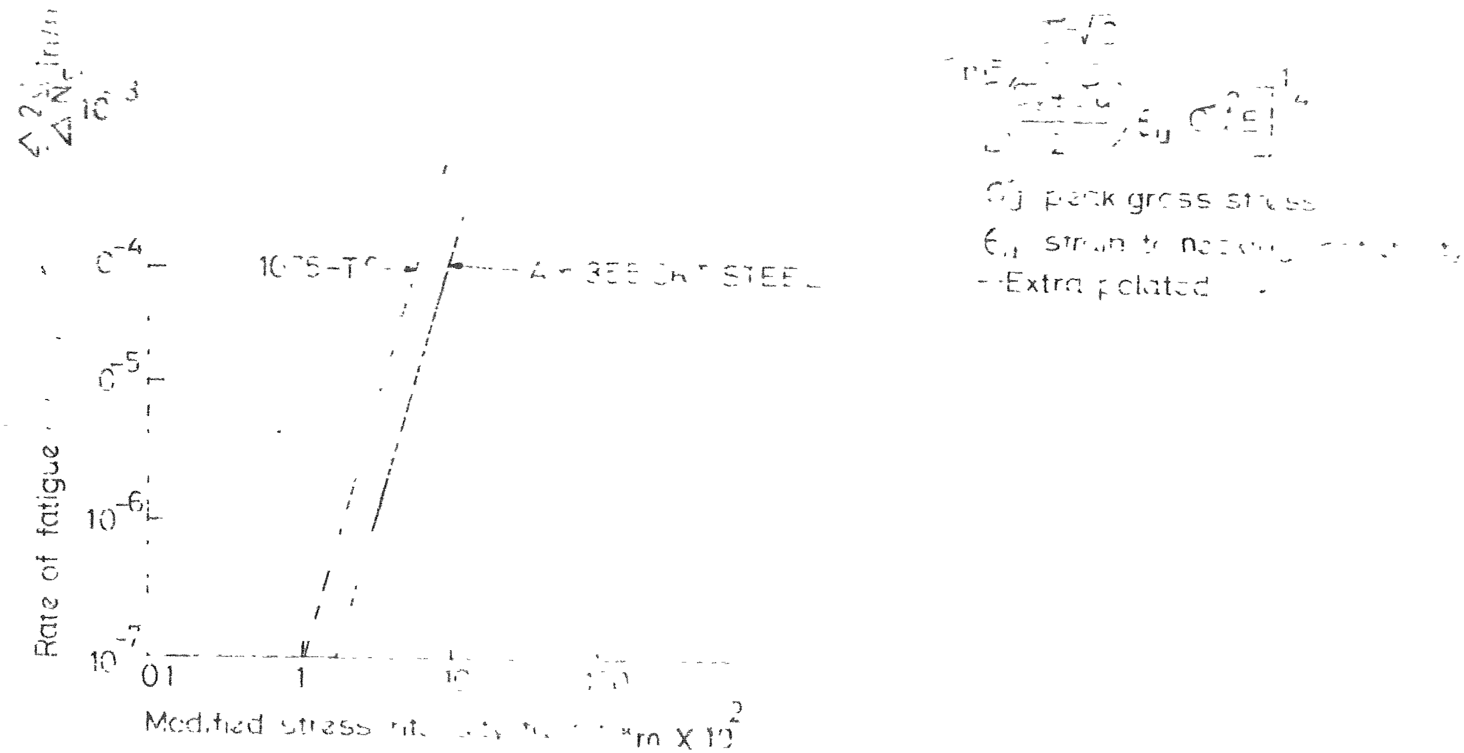


FIG 3.4 VARIATION OF RATE OF CRACK PROPAGATION AS AFFECTED BY MODIFIED STRESS INTENSITY FACTOR.

The computed values of β according to Eqn. 9.3 are given in the last column of Table 9.1.

9.3.4 Design Procedure with Emphasis on Material Choice

The stress analysis of the problem of part-through crack in a curved sheet is not yet solved. The corresponding solution for a flat sheet relating the flaw shape to the fracture stress with correction for plasticity at the crack tip is given by Tiffany and Masters (65). The fracture toughness K_c for the case of part-through crack is given by

$$K_c = 1.1 \sqrt{\pi} \quad \sigma \sqrt{\frac{a_{cr}}{Q_{cr}}} \qquad .. (9.4)$$

where K_c = fracture toughness

σ = applied stress

a_{cr} = flaw depth at the onset of fracture

Q_{cr} = flaw shape parameter at the onset of fracture and is given by

$$Q = \phi^2 - 0.212 \quad (\frac{\sigma}{\sigma_y})^2 \qquad .. (9.5)$$

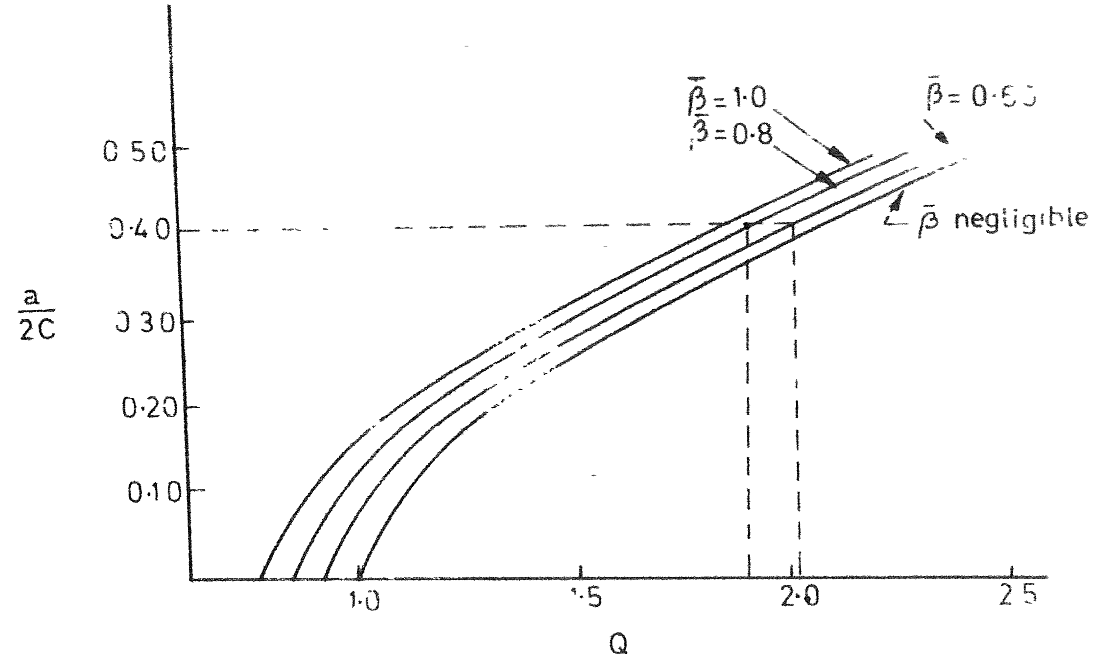
where ϕ = elliptic integral of the second kind

$$= \int_0^{\pi/2} \sqrt{\{1 - (\frac{a^2-c^2}{a^2}) \sin^2 \theta\}} \, d\theta \qquad . (9.6)$$

a = depth of the flaw

c = semimajor axis of the flaw

The flaw shape parameter curves relating Q and (a/2c) are plotted for various stress levels (65) and are reproduced in Fig. 9.5. Knowing the fracture toughness of a material Eqn. 9.4 enables one to know the critical crack size that leads to fracture at a given stress level. For the various materials considered in Table 9.1, the relationship between stress σ and the critical crack size calculated according to Eqn. 9.4 are plotted and are shown in Figs. 9.6 to 9.11. To proceed further with the design, an estimate of the initial crack size is required. The most effective cracks in the cylindrical vessel are those which are located longitudinally on the shell so that they open out under hoop tension due to internal pressure. Consequently, it is assumed that the critical crack to be considered is under tension due to hoop stress. To estimate the initial crack size, proof loading upto 90% of the yield stress level is resorted to and if the vessel has not failed during the loading, one can infer that the initial crack size is less than the value a_{cr}/Q_{cr} corresponding to the 90% stress level on the curve relating stress and critical crack size for a material. As a conservative estimate, the value of a_{cr}/Q_{cr} corresponding to 90% of the yield stress can be taken as the initial crack size. Since, the working stress is to be less than 60% of the



$Q = \left[\phi^2 - 0.212 \left(\frac{\sigma}{\sigma_y}\right)^2\right]$

Q = Flaw shape parameter

ϕ = Complete elliptic integral of the second kind

σ = Gross applied stress

σ_y = Yield stress (0.2 % offset)

$\bar{\beta} = \sigma / \sigma_y$

FIG. 9.5 FLAW SHAPE PARAMETER CURVES FOR PART-THROUGH CRACKS (65)

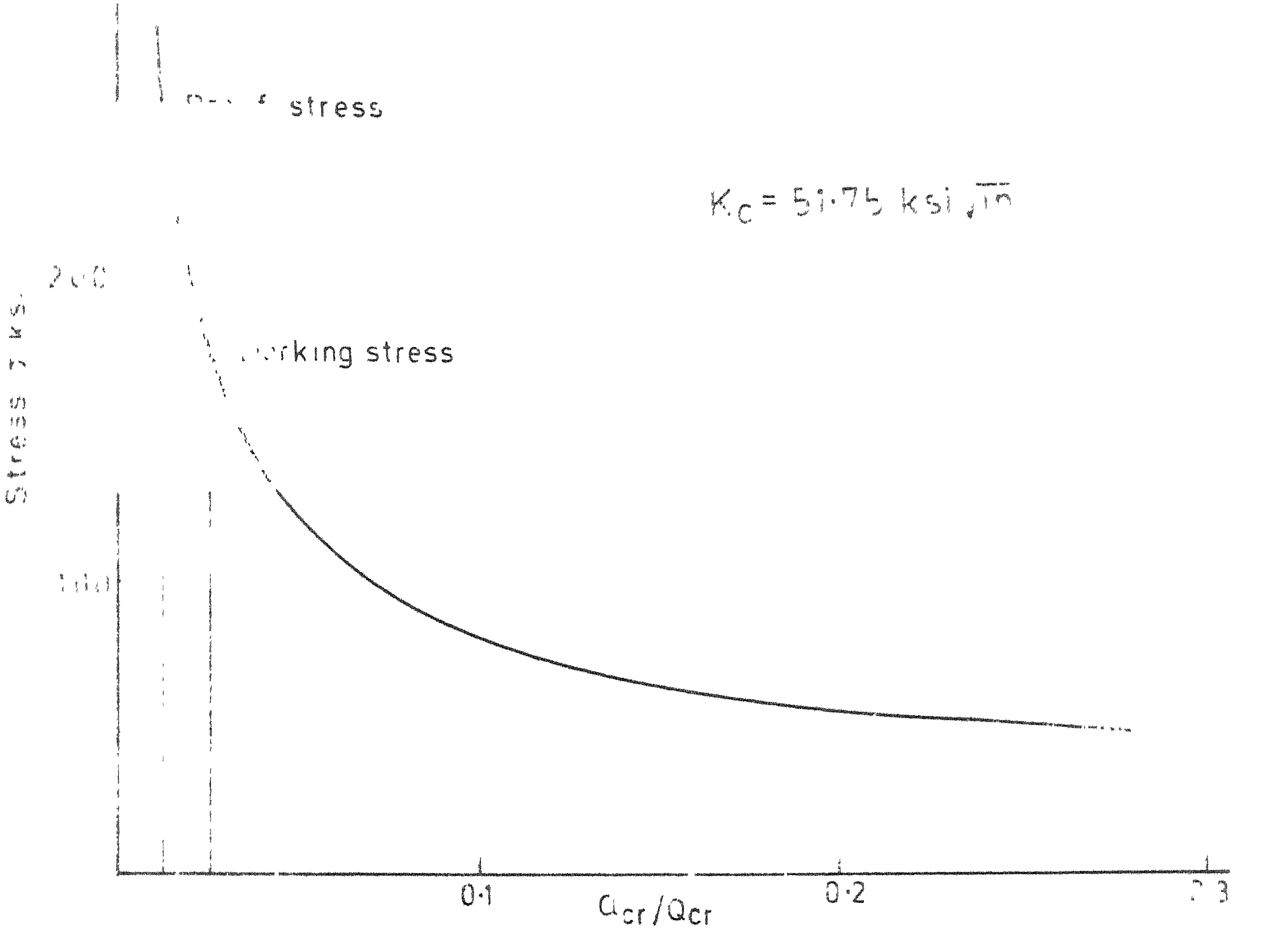


FIG. 9·6 RELATIONSHIP BETWEEN STRESS AND CRITICAL CRACK SIZE FOR MATERIAL 1

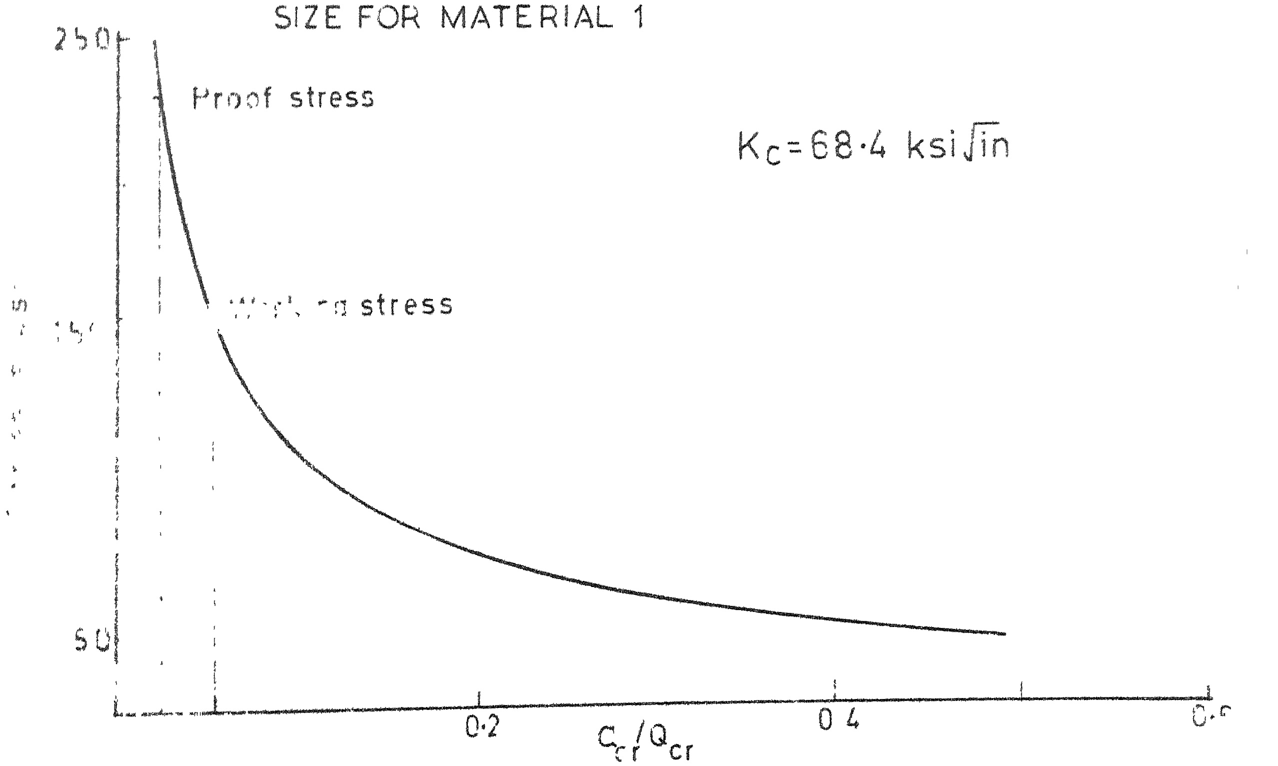


FIG. 9·7 RELATIONSHIP BETWEEN STRESS AND CRITICAL CRACK SIZE FOR MATERIAL 2

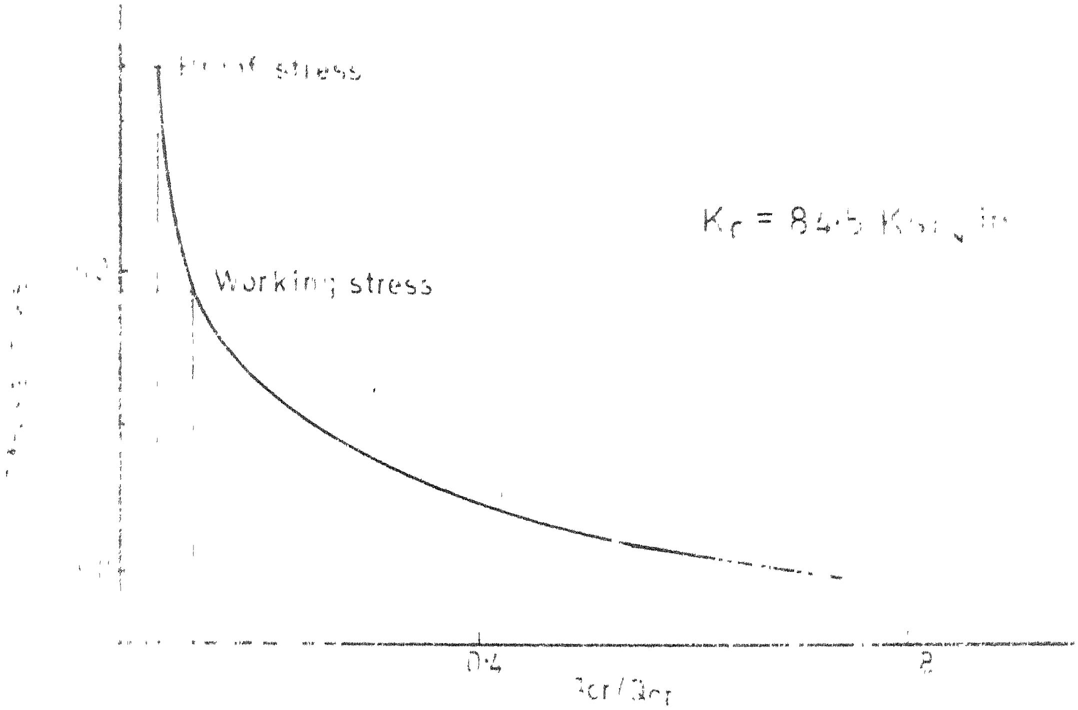


FIGURE 8 RELATIONSHIP BETWEEN STRESS AND CRITICAL CRACK SIZE FOR MATERIAL 3

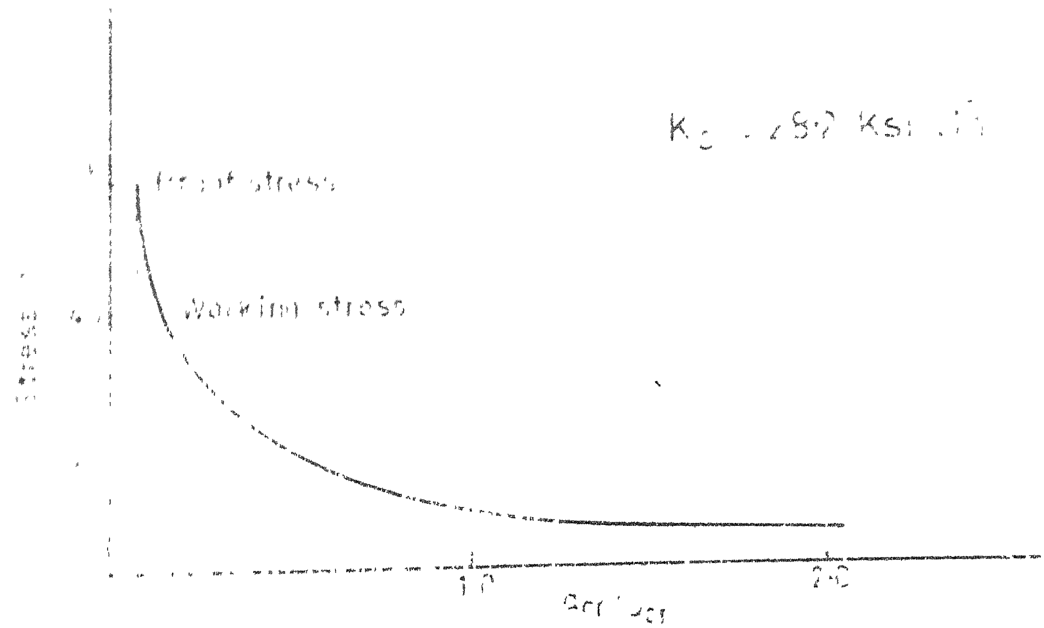
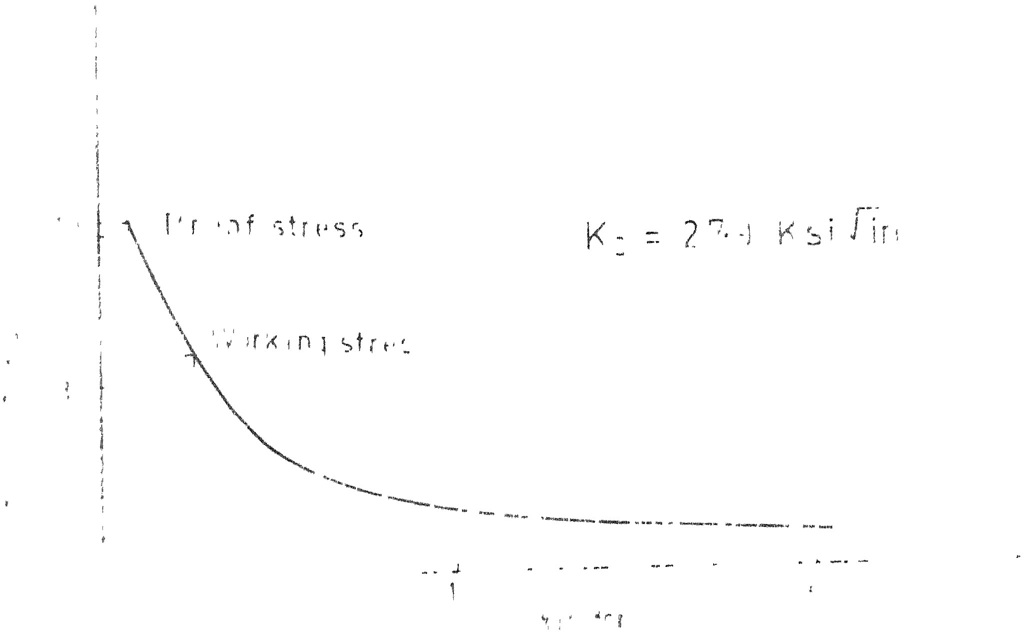
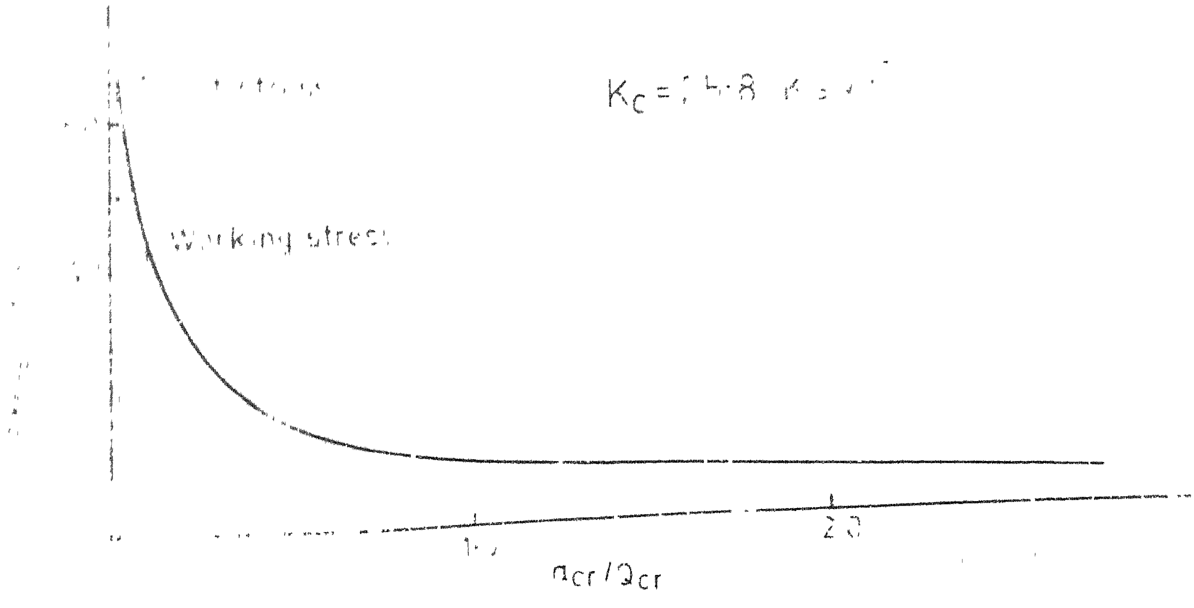


FIGURE 9 RELATIONSHIP BETWEEN STRESS AND CRITICAL CRACK SIZE FOR MATERIAL 4



[illegible] RELATIONSHIP BETWEEN STRESS AND CRITICAL CRACK SIZE FOR MATERIAL 5



[illegible] RELATIONSHIP BETWEEN STRESS AND CRITICAL CRACK SIZE FOR MATERIAL [illegible]

considered. Since the nominal working stress should not exceed 60% of the yield stress the thickness 't' of the vessel is fixed on the basis

$$t = \frac{p\,D}{2 \times 0.6\sigma_y} \qquad .. (9.7)$$

where t = thickness in inches

p = internal pressure = 2000 psi

σ_y = yield stress

D = diameter = 50"

The values of critical crack sizes corresponding to 90% and 60% of the yield stress level corresponding to various materials are obtained from Figs. 9.6 to 9.11. Table 9.2 gives the details of working stress, proof stress, the thickness of the vessel wall according to Eqn. 9.7 and the values of a_{cr}/Q_{cr} at working and proof stress levels. Corresponding to the ratio of a/2c as assumed i.e., 0.4 the values of Q_{cr} are determined from the Fig. 9.5 giving the details of flaw shape parameter curves at various stress levels. Knowing the values of Q_{cr} corresponding to proof and working stress levels, i.e. the initial and final values of a_{cr}, from Fig. 9.5 the individual values of $a_{initial}$, $2c_{initial}$ and a_{final}, $2c_{final}$ are determined and are given in Table 9.3. The values of the modified stress intensity factor corresponding to 60% working stress

TABLE 9.2: PRESSURE VESSEL DESIGN DETAILS (SAFE LIFE DESIGN)

Material	Working stress 0.6 σ_y ksi	Proof stress 0.9 σ_y ksi	Vessel walls thickness 't' inches	a_{cr}/Q_{cr}	
				Proof stress level	Working stress level
1	171	256	0.292	0.012	0.026
2	155.5	233	0.321	0.026	0.052
3	145	218	0.344	0.04	0.08
4	38.6	57.7	1.295	0.08	0.16
5	34.6	51.9	1.445	0.08	0.26
6	43.9	66.6	1.14	0.02	0.10

TABLE 9.3: CRACK DIMENSIONS AND RATES OF PROPAGATION

Material	$a_{initial}$ inches	a_{final} inches	$2c_{initial}$ inches	$2c_{final}$ inches	Modified stress Intensity factor Km	Rate of crack propagation	Service life cycles
1	0.0224	0.0527	0.056	0.132	1.5×10^{2}	$\simeq10^{-7}$	76×10^{4}
2	0.0485	0.1055	0.121	0.262	2.12×10^{2}	6.5×10^{-6}	2.7×10^{4}
3	0.0745	0.162	0.1865	0.405	2.4×10^{2}	4×10^{-6}	5.49×10^{4}
4	0.1495	0.324	0.374	0.81	12.9×10^{2}	10^{-1}	very low
5	0.1495	0.527	0.374	1.29	6.05×10^{2}	1.5×10^{-4}	6.3×10^{4}
6	0.0374	0.2027	0.0935	0.509	0.6×10^{2}	$\simeq10^{-7}$	415×10^{4}

level are determined from Eqn. 9.2 and the corresponding rates of crack propagation are obtained from Fig. 9.4. The values of K_m and the rate of crack propagation $\frac{\Delta 2c}{\Delta N_S}$ are given in Table 9.3. The number of cycles required to grow the crack from $2c_{initial}$ to $2c_{final}$ is determined and is given in the last column of Table 9.3. The procedure is repeated for various materials and the final details are given in Table 9.4, along with the weights and material costs computed using the properties given in Table 9.1. The utility U_R calculated according to Eqn. 9.1, for the various materials' choice is also given, in the last column of Table 9.4. It can be seen from a comparison of the net returns from various design decisions, the materials one and six yield a "profit", while the rest involve a "loss". The 4th material is not evaluated for its utility since the service life is too low. Obviously material six is the best choice as it yields maximum return and satisfies all other design criteria. As could be seen from Table 9.3 the crack depths at fracture given by a_{final} are less than the wall thicknesses in all the cases indicating that fracture occurs without any prior indication. For the optimal design chosen viz, using material six with wall thickness 1.14 inches, the vessel should be put out of service before 415 x 10^4 cycles unless its further life can be

TABLE 9.4: DESIGN SOLUTIONS USING VARIOUS MATERIALS (SAFE LIFE DESIGN)

Material	Vessel Wall Thickness 't' inches	Weight of the vessel w per inch length lbs	Material cost C per inch length	Service life N cycles	Net return U_R Rs.
1	0.292	13.0	39	76×10^4	243000
2	0.321	14.3	57.2	2.7×10^4	-47450
3	0.344	15.3	76.5	5.44×10^4	-57500
4*	1.295	21.8	43.6	< 10	-
5	1.495	24.2	48.4	6.3×10^4	-32000
6	1.14	19.2	38.4	415×10^4	1248600

* Not considered in comparative evaluation since the service life is very low.

ensured by suitable nondestructive tests and direct flaw observations.

9.3.6 Material Choice for Fail-Safe Design

While designing for safety at the onset of fracture keeping all the conditions of design as above, viz, the working and proof stress levels and the materials, the shell diameter cannot be a priori fixed and has to be fixed along with the thickness so that the flaw penetrates through the full depth before fracture occurs. The quantity (a_{cr}/Q_{cr}) initial as well as final, in Table 9.2 remains valid even in this case except that the wall thickness should not be more than a_{cr} so that, the crack fully propagates through the wall before the vessel fractures; i.e.

$$a_{cr} = t \quad \text{for fracture safe design} \qquad .. (9.8)$$

where t = wall thickness

a_{cr} = flaw depth at fracture

Since the value of Q_{cr} is known as before, from the Fig. 9.5 giving the flaw shape parameter curves for working stress level, t is computed for various materials. The final values of (a_{cr}/Q_{cr}) and the computed vessel thickness for various materials are given in Table 9.5. Since the working stress level is fixed as 60% of the yield stress, the corresponding diameter D of the vessel

TABLE 9.5: DESIGN SOLUTIONS USING VARIOUS MATERIALS (FAIL - SAFE DESIGN)

Material	$\frac{a_{cr}}{\hat{a}_{cr}}$	thickness t inches	Diameter D inches	Service life N cycles	Weight W lbs/ in length	Cost C Rs. per in length	Utility Net return Rs.
1	0.026	0.053	8.55	76×10^{4}	0.40	1.2	376800
2	0.052	0.105	16.3	2.7×10^{4}	1.53	6.12	6980
3	0.0[illegible]	0.162	23.8	5.44×10^{4}	3.42	17.10	8300
4*	0.16	0.324	12.5	< 10	-	-	-
5	0.26	0.53	18.3	6.3×10^{4}	3.23	6.46	22940
6	0.10	0.202	8.9	415×10^{4}	0.60	1.2	2038800

* Not considered in comparative evaluation since the service life is very low

is calculated, knowing the vessel thickness from the equation

$$D = \frac{2.\ t.\ 0.6\ \sigma_y}{2000} \quad \text{.. (9.9)}$$

where D = diameter in inches

t = thickness in inches

σ_y = the yield stress in psi

The value of the diameter so calculated for various materials is also given in Table 9.5. Since the materials, stress levels and flaw shape are the same as in the earlier case of safe life design, the service lives as calculated previously remain unaltered even in the case of fail safe design. The weight and cost corresponding to the design solutions along with the associated utilities computed according to Eqn. 9.1 are also given in Table 9.5. Comparing the utilities corresponding to various design solutions, once again it can be noted that the sixth material is the optimal as it yields maximum utility or return.

9.4 DISCUSSION

The method of optimal material choice for fracture resistant design discussed in this chapter enables one to choose a material out of an available set of materials so that the utility of design solution is maximized.

Even though the utility is considered synonymous with net monetary return from service life, cost incurred due to weight and initial cost, any alternative definition could be used if necessary for the objective function wherein more design criteria like reliability and deformation could also be included. It is worth noting that the optimality criterion and design variables considered in the present problem are different from those of Heer and Yang (127), wherein they have attempted to find an optimum proof load level and the probability of failure so that the weight of the structure is minimized subject to total cost and reliability constraints. Also they consider the design problem with a single material, in contrast to the present problem where the number of materials are considered and proof load level is a priori fixed.

Even though some of the material properties assumed in the problems solved and the effects of curvature of the vessel on crack propagation are considered negligible because of lack of information, the method discussed remains valid and can be used if more accurate information regarding the material properties and stress analysis of part-through crack on a curved surface are available. For example, fatigue failure is not

considered in the design of the vessels; if data regarding the number of cycles the material can withstand at various stress levels is available, the same should also be considered in the service life estimation.

CHAPTER TEN

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

10.1 SUMMARY AND CONCLUSIONS

In this investigation, some selected problems in the area of statistical aspects of size effects on strength and fracture of materials (i.e. brittle fracture statistics) are discussed. The object and scope of the thesis is presented in Section 1.5. A detailed discussion as relevant to the various problems studied, is given at the end of each chapter. In the following a summary and conclusions based on the present investigation are given, while potential areas of further investigation are given in Section 10.2.

A critical review of the various statistical strength theories as applied to various materials has indicated the inadequacy of the existing approaches in characterizing the size dependent scatter in materials. The possibility of an alternative approach viz, constructing a strength distribution function suitable to a given material, as given by its size-mean strength relation is investigated. Following the work of Tsai and Kolsky (87), the aforementioned problem

is formulated using the weakest link concept and a distribution function is obtained corresponding to the size-mean strength relation of Weibull in a reparameterized form. Essentially the method of approach currently suggested is converse in nature to the existing methods, since the strength distribution function is obtained corresponding to the size-mean strength relation of a material rather than assuming a priori the distribution function. Even though the strength distribution function is obtained in closed form for the case studied in the thesis, treatment of more general forms of size-mean strength relations needs numerical techniques.

Even though concrete is not a material that obeys the weakest link concept, the pattern of decrease in mean strength and scatter with increase in specimen size suggests an idealization of the material as that obeying weakest link concept, for the purpose of characterization of scatter in strength. Using the test results of Kadleček and Špetla (43), an attempt is made to characterize the size dependent scatter in direct tensile strength of concrete, by choosing the material parameters in the size-mean strength relation by trial and error. It is noticed that the experimental and theoretical strength distribution functions are in agreement with

varying degrees corresponding to different specimen sizes, indicating that the same material parameters are inadequate to characterize the scatter with equal accuracy at various specimen volumes.

Subsequently some possible practical applications of statistical strength theories are studied. Because of the existence of size dependent scatter in strength of materials, a problem of interest is to study the minimum number of specimens required to be tested (as affected by specimen size) so that the mean strength of the material is predicted with a given accuracy and reliability. Using the test results of Kadleček and Špetla and applying Student's t distribution, an attempt is made to study the variation in the minimum number of specimens to be tested (for a given error of five percent with ninety percent probability, in predicting the mean direct tensile strength of concrete), with the specimen volume. As it is observed that smaller number of larger specimens are required, an attempt is made to find the optimum specimen size and sample number so that the total cost of testing is a minimum. After a study of some test results, it is found that more number of tests are required to predict tensile strength of concrete, than the compressive strength.

A unified phenomenological theory of size effects on strength, stiffness and ductile-brittle transition in concrete-like materials is developed. Concrete-like materials are considered to be two phase composite materials, with one of the phases being voids due to microcracking. An estimate of the voids due to microcracking is made in terms of the stress and size dependent probability of failure. Using an approximate expression for the effective Young's modulus of a two phase composite material in the presence of voids, an expression for the size and stress dependent Young's modulus is derived. The expression and the underlying theory are shown to predict higher stiffness and greater ductility associated with specimens of smaller volume. The predictions are observed to be qualitatively in agreement with experimental observations in which steel fiber reinforced cement mortar briquettes of different sizes are tested in tension.

As an application of fracture mechanics in engineering design, the design problem of a pressure vessel in the presence of a part-through crack with emphasis on material choice is considered. Six candidate materials are considered. The utility of the design decision or material choice is considered to be

same as the net monetary return, viz, the return from service life, less the cost of material and the cost incurred due to its weight in service life.

10.2 RECOMMENDATIONS FOR FURTHER RESEARCH

The formulation and the method of solution of the problem of size effects on strength of materials discussed in the thesis, needs further theoretical studies regarding the uniqueness of distribution functions corresponding to a given size-mean strength relation. Also closed form solutions and numerical solutions to obtain the distribution functions relevant to various forms of size-mean strength relationships (as suitable to different materials) are to be studied in detail so that the present formulation can be applied for different materials for engineering use.

Analytical and experimental studies on various materials for finding the size dependent coefficient of variation under general stress states would be of value in standardizing testing methods. Such studies in conjunction with the studies of actual mechanisms of failure would also enable expressing the scatter in the strength of materials in terms of the characteristic material parameters.

The problem of the effective elastic moduli of

a composite material in the presence of voids, (in particular interface discontinuities between various phases) will be helpful in studying materials like concrete in a more rational manner. Also experimental studies on such materials for size effects and volume changes will be valuable in furthering the knowledge of the mechanics of such materials.

The problem of a part-through crack in a curved sheet for stress analysis as well as for rate of crack propagation would be of immense value in engineering design of pressure vessels etc.

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